Faculty of Mathematical Studies

MA204 Real Analysis: Uniform Convergence

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Introduction: Aims

In dealing with sequences and series of functions a number of unexpected phenomena occur. We shall see examples where a sequence of continuous functions converges to a discontinuous function; where a sequence of integrable functions converges to a function which is not integrable; where the integral of a function represented by an infinite series is different from the sum of the integrals of the individual terms in the series.

Our aim is to investigate circumstances under which we can be sure that such undesirable behaviour will not occur. This involves looking at convergence in a new way, leading to the idea of uniform convergence.

The ideas can be formulated in the general setting of functions on metric spaces. The mathematics is no different from that for real-valued functions of a real variable. Nevertheless we shall restrict ourselves to functions $f:\mathbf{R}\to\mathbf{R}$ in order to keep abstraction to a minimum. Those of you who are so inclined can translate most of the topic to deal with functions from an arbitrary set X to a metric space (Y,d).

Prerequisites

MA201 work on convergence of sequences and series. In particular you should revise the ratio test for convergence, and the idea of radius of convergence for a power series. You should also know about convergent GPs.

Click **here** for some revision notes on power series.

Click **here** for some revision notes on GPs

Some Strange Examples

Example 1

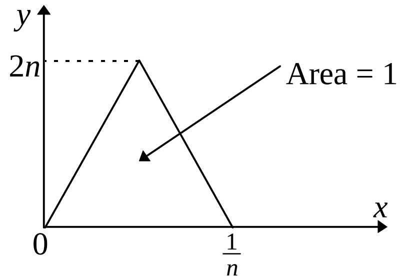
We consider the sequence of functions defined on [0,1] by $f_n(x) = x^n$.
For $0 \le x < 1$, $f_n(x) \to 0$, whereas $f_n(1) \to 1$.

(For any $\varepsilon > 0$, no matter how small, verify that there is a positive integer N such that for some $x > 1 - \varepsilon$, $f_n(1) - f_n(x) > 0.999$ for all $n > N$. Verify that 0.999 could be replaced by any number $1 - \delta$, where $\delta > 0$. The value of N will depend on the numbers δ and ε.) Click **here** for details.

So this provides an example of a sequence of continuous functions where the limit function exists but has a discontinuity.

Example 2 On the interval [0,1] we define a sequence of functions as follows:

$$f_n(x) = \begin{cases} 0 & \text{if } x \ge \frac{1}{n}, \\ 4n - 4n^2 x & \text{if } \frac{1}{2n} \le x < \frac{1}{n}, \\ 4n^2 x & \text{if } 0 \le x < \frac{1}{2n}. \end{cases}$$



Every member of the sequence is continuous, and for each value of x the sequence converges to the zero function f. So Every member of the sequence is Riemann integrable, and so is the limit function. For every n the area of the triangle in the diagram is 1, so we have $\lim \int_0^1 f_n = 1$, but $\int_0^1 \lim(f_n) = 0$.

Example 3 We define a sequence of functions by $f_n(x) = \dfrac{\sin nx}{n}$. So for every value of x the sequence converges to the zero function f. Therefore $f'(x) \equiv 0$. Now we know that $f_n'(x) = \cos nx$, which does not tend to zero for every x. For example $f_n'(2\pi) = 1$ for all n, and $f_n'(\pi) = (-1)^n$. So for some x the sequence of derivatives has a limit, and for other values of x it does not. Even where there is a limit, this example shows that $\lim_{n\to\infty} \dfrac{df_n(x)}{dx}$ and $\dfrac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right)$ need not be the same.

Pointwise Convergence and Uniform Convergence

Definition 1 A sequence of functions $(f_n(x))$ is said to converge pointwise to a function $f(x)$ for x belonging to some set A if for each value of x in A $(f_n(x))$, considered as a sequence of real numbers, converges to the real number $f(x)$. Symbolically:

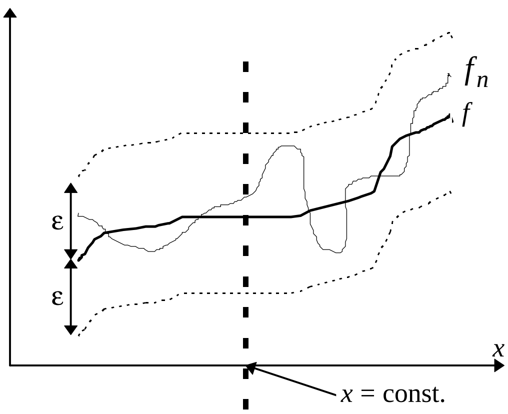
$$\forall \varepsilon > 0, \forall x \in A, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \varepsilon.$$

Definition 2 A sequence of functions $(f_n(x))$ is said to converge uniformly to a function $f(x)$ for x belonging to some set A if

$$\forall \varepsilon > 0, \exists N, \forall x \in A, \forall n \geq N, |f_n(x) - f(x)| < \varepsilon.$$

The crucial difference between these definitions relates to the order of quantification. In the first case the value of N can vary with x as well as ε, whereas in the second case N must exist independent of x. (Click **here** for a simple example on order of quantification.)

It is helpful to think about the two definitions graphically. In relation to the limit function f we need to think about an "ε-strip" around the graph of f. For a sequence of functions to converge uniformly, given a positive number ε, for sufficiently large n the entire graph of $f_n(x)$ must lie within the "ε-strip".



In the case of pointwise convergence we take a cross section through the set of graphs of $f_n(x)$ and f with a line x = constant. This then gives a sequence of numbers (the y values corresponding to this fixed value of x) which for sufficiently large n lie in the "ε-strip". However this does not have to happen simultaneously for all values of x. A good way to understand the difference is in relation to examples 1 and 2, where the sequence of functions converges pointwise to zero in each case, but convergence is not uniform.

Note that uniform convergence on a set A implies pointwise convergence, but not conversely.

Theorem 1

Let $(f_n(x))$ be a uniformly convergent sequence of *continuous* functions with limit function $f(x)$ for x belonging to some set A. Then $f(x)$ is continuous on A.

Proof

We show that $f(x)$ is continuous at $x = a$ for each $a \in A$. Let ε be an arbitrary positive number. Because of uniform convergence,

$$\exists N, \forall x \in A, \forall n \geq N, |f_n(x) - f(x)| < \varepsilon/3.$$

Now f_N is continuous at a so

$$\exists \delta, |x-a| < \delta \Rightarrow |f_N(x) - f_N(a)| < \varepsilon/3.$$

We therefore conclude, using the triangle inequality, that for $|x-a| < \delta$,

$$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

We can use this as a way of showing that convergence is non-uniform,
as with **Example 1** where the limit function has a discontinuity.

In the context of metric spaces we can think of uniform convergence as convergence using the supremum metric for example 8 in the notes on metric spaces, and we express this as a theorem.

Theorem 2

A sequence $(f_n(x))$ of bounded functions converges uniformly to the function f on the set A if and only if the sequence of real numbers

$$K_n = \sup\{|f_n(x) - f(x)| : x \in A\} \to 0 \text{ as } n \to \infty.$$

Proof

$$\begin{aligned}
&K_n \to 0 \Leftrightarrow \forall \varepsilon > 0, \exists N, \forall n \geq N, |K_n| < \varepsilon \\
&\Leftrightarrow \forall \varepsilon > 0, \exists N, \forall n \geq N, \sup\{|f_n(x) - f(x)| : x \in A\} < \varepsilon \\
&\Leftrightarrow \forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x \in A, |f_n(x) - f(x)| < \varepsilon \\
&\Leftrightarrow f_n \to f \quad \text{uniformly in } A.
\end{aligned}$$

Theorem 3 (The Cauchy Criterion)

A sequence $\left(f_n(x)\right)$ of bounded functions converges uniformly to the function f on the set A if and only

$$\forall \varepsilon > 0, \exists N, \forall m, n \geq N, \forall x \in A, \left| f_m(x) - f_n(x) \right| < \varepsilon.$$

Proof

Suppose $\left(f_n(x)\right)$ converges uniformly to f on the set A. Then, given $\varepsilon > 0$,

$\exists N, \forall n \geq N, \forall x \in A, \left| f_n(x) - f(x) \right| < \frac{\varepsilon}{2}$. Using the triangle inequality then gives

$\exists N, \forall n \geq N, \forall x \in A, \left| f_m(x) - f_n(x) \right| \leq \left| f_m(x) - f(x) \right| + \left| f_n(x) - f(x) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Conversely, suppose the Cauchy Criterion is satisfied. Then for each x, the sequence $\left(f_n(x)\right)$ is a Cauchy Sequence of real numbers, and therefore has a limit, which we denote by $f(x)$. Thus we have pointwise convergence, but we must demonstrate uniform convergence.

Given $\varepsilon > 0$, the Cauchy criterion tells us that

$\exists N, \forall m, n \geq N, \forall x \in A, \left| f_m(x) - f_n(x) \right| < \frac{\varepsilon}{2}$. Now $f_n(x) \to f(x)$ pointwise, and so

$\forall x \in A, \exists N_x \geq N, \forall n \geq N_x, \left| f_n(x) - f(x) \right| < \frac{\varepsilon}{2}$. We deduce that

$\forall x \in A, \forall n \geq N, \left| f_n(x) - f(x) \right| \leq \left| f_n(x) - f_{N_x}(x) \right| + \left| f_{N_x}(x) - f(x) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$

Thus convergence is uniform.

Uniform Convergence of Infinite Series

As with numerical series, we can define convergence through the sequence of partial sums. So given a sequence of functions $\left(f_n(x)\right)$ defined on some set *A*, we define $S_n(x) = \sum_{k=1}^{n} f_k(x)$, and we say that the series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly with sum $f(x)$ when the sequence $\left(S_n(x)\right)$ converges uniformly to $f(x)$.

Theorem 4 If the series $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly to $f(x)$ on a set *A*, and if for each *n*, f_n is continuous on *A*, then *f* is continuous on *A*.

Proof Apply **Theorem 1** to the sequence of partial sums. Click **here** for details.

Example 4 We consider the exponential series $\exp(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$

Now for $x \in [-R, R]$ we have, using the notation above for partial sums,

$\left|\exp(x) - S_n(x)\right| \le \sum_{k=n+1}^{\infty} \frac{|x|^k}{k!} \le \sum_{k=n+1}^{\infty} \frac{R^k}{k!}$. The latter is a real number, K_n, which tends to zero. Hence by **Theorem 2** convergence is uniform for $x \in [-R, R]$.

Now we shall show that convergence is not uniform on **R**. We have, for $x > 0$,

$$\exp(x) - S_n(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \ge \frac{x^{n+1}}{(n+1)!} > 1 \text{ if } x > n+1.$$

So $\forall N, \exists n \ge N, \exists x \in \mathbf{R}, \exp(x) - S_n(x) > 1$. Thus convergence is not uniform.

This example emphasises that uniformity of convergence depends on the domain.

Other examples will illustrate this, so that in **Example 1** if we take an interval such as [0,0.5] in place of [0,1] we find that convergence is uniform on the smaller interval. Click **here** for details.

Theorem 5 (The Weierstrass M-test)

Suppose that $(f_n(x))$ is a sequence of functions defined on a set A, and that (M_n) is a sequence of real numbers with the properties that $\forall n, \forall x \in A, |f_n(x)| \le M_n$ and that $\sum_{k=1}^{\infty} M_k$ converges. Then the series $\sum f_n(x)$ converges uniformly on A.

Proof

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$; $T_n = \sum_{k=1}^{n} M_k$. Because the sequence (T_n) converges it is a Cauchy sequence. Using the condition $\forall n, \forall x \in A, |f_n(x)| \le M_n$ then tells us that the sequence $(S_n(x))$ satisfies the **Cauchy criterion**, so that the series of functions converges uniformly on A.

Exercise 1 Prove Theorem 5 directly from Theorem 2 or from the definitions of convergence, without using the Cauchy criterion. Click **here** for a solution.

Theorem 6

Suppose that $\sum a_n x^n$ is a power series with radius of convergence R. Let K be any positive number satisfying $K < R$. Then the power series converges uniformly on the interval $[-K, K]$.

Proof

For all $x \in [-K, K], |a_n x^n| \le |a_n| K^n \ (= M_n)$. The series $\sum |a_n| K^n$ converges, and so using the M-test proves the result.

Corollary

The sum function for a power series is continuous at all points inside the circle of convergence.

Proof

Suppose $|x_0| < R$. Choose K to satisfy $|x_0| < K < R$. Convergence is uniform on $[-K, K]$, and so the sum function is continuous on $[-K, K]$. In particular the sum function is continuous at x_0.

Some Interchange Theorems

In this section we look at some results which enable us, for example, to interchange the order of summation and integration without changing the answer.

Theorem 7 Let $(f_n(x))$ be a sequence of Riemann integrable functions converging uniformly to a function f on $[a,b]$. Then f is integrable on $[a,b]$ and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n \quad \text{i.e.} \quad \int_a^b \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_a^b f_n.$$

Proof

$\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x \in [a,b], \left| f_n(x) - f(x) \right| < \frac{\varepsilon}{b-a},$

i.e. $\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x \in [a,b],\ f_n(x) - \frac{\varepsilon}{b-a} < f(x) < f_n(x) + \frac{\varepsilon}{b-a}.$

Since the f_n, and therefore also f, are bounded on $[a,b]$, it follows that for any subset S of $[a,b]$,

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \sup_{x\in S}\left(f_n(x)\right) - \frac{\varepsilon}{b-a} \leq \sup_{x\in S}\left(f(x)\right) \leq \sup_{x\in S}\left(f_n(x)\right) + \frac{\varepsilon}{b-a}.$$

Now let α be any subdivision of $[a,b]$, and consider the upper sums. The inequality above tells us that $\forall \varepsilon > 0, \exists N, \forall n \geq N, M_i(f_n) - \frac{\varepsilon}{b-a} \leq M_i(f) \leq M_i(f_n) + \frac{\varepsilon}{b-a}$ for all i. We conclude that $\forall \varepsilon > 0, \exists N, \forall n \geq N, S(\alpha, f_n) - \varepsilon \leq S(\alpha, f) \leq S(\alpha, f_n) + \varepsilon.$

Now $\int_a^b f_n = \inf_{\alpha}\{S(\alpha, f_n)\}$, and so we conclude that

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \int_a^b f_n - \varepsilon \leq \inf_{\alpha}\{S(\alpha, f)\} \leq \int_a^b f_n + \varepsilon, \quad \text{i.e.} \quad \lim_{n\to\infty}\int_a^b f_n = \inf_{\alpha}\{S(\alpha, f)\}.$$

Click **here** for a more detailed discussion of the latter part of this proof.
By considering lower sums (click **here** for details) we show that $\lim_{n\to\infty}\int_a^b f_n = \sup_{\alpha}\{s(\alpha, f)\}$. The result follows.

Theorem 8 Let $(f_n(x))$ be a sequence of Riemann integrable functions, and suppose that the series $\sum f_n(x)$ converges uniformly to a function f on $[a,b]$. Then f is integrable on $[a,b]$ and $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$. This can be written as $\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n$. We say that the series can be integrated term-by-term.

Proof Apply **Theorem 7** to the sequence of partial sums. Click **here** for details.

We can use this result to demonstrate results concerning elementary functions, taking the point of view that they are defined as power series. *Example on next page.*

Example 5
We showed in **Example 4** that the exponential series converges uniformly in any bounded closed interval, for example $[0,a]$. So the last corollary tells us that

$$\int_0^a \exp(x) = \int_0^a \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \int_0^a \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{a^{n+1}}{(n+1)!} = \sum_{k=1}^{\infty} \frac{a^k}{k!} = \exp(a) - 1.$$

The next theorem concerns differentiability. Note that in **Example 3** the sequence of functions converges uniformly, but this still does not ensure that the derivative of the limit is the limit of the derivatives.

Theorem 9
Let $(f_n(x))$ be a sequence of differentiable functions, with continuous derivatives, converging pointwise to a function f on an open interval I (finite or infinite). Suppose that the sequence (f_n') of derivatives converges uniformly on I to a function g. Then f is differentiable and $f' = g$.

Proof
Let a denote a (fixed) point of I, and let x denote another (variable) point of I. We then have

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)dt.$$

Letting n tend to infinity and using **Theorem 7** then gives

$$f(x) = f(a) + \int_a^x g(t)dt.$$

Since g is a uniform limit of continuous functions it is continuous, and so we deduce that f is differentiable and that $f'(x) = g(x)$ for all x in the interval I.

Exercise 2 Formulate and prove a corollary for infinite series. Click **here** for details.

Theorem 10 Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R.

Then for all $x \in (-R, R)$, f is differentiable at x and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. We say that the power series can be differentiated term-by-term.

Proof

From MA201, the two series have the same radius of convergence. We assume that R is positive. Given $x \in (-R, R), \exists b, |x| < b < R$. By **Theorem 6** the power series converges uniformly on $[-b, b]$, and so the result follows from the exercise following **Theorem 9.**

Some revision notes on power series

Definition A power series is an infinite series of the form $\sum_{n=0}^{\infty} a_n x^n$. For this course the coefficients a_n and the variable x are real, but in general (for example in MA202) they can be complex, and in fact the analysis is very similar in both cases. The coefficient a_0 is called the constant term.

The basic theorems about power series are as follows. You may have met some of them in MA201 or in MA202

Theorem A If a power series converges for $x = x_0$ then it converges absolutely for $|x| < |x_0|$.

Theorem B If a power series diverges for $x = x_0$ then it diverges for $|x| > |x_0|$.

Theorem C For any power series one of the following three possibilities occurs:
(i) it converges only for $x = 0$. e.g. $\sum n!x^n$.
(ii) it converges for all x. e.g. $\sum \frac{x^n}{n!}$.
(iii) there is a positive real number R such that the power series converges absolutely for $|x| < R$ and diverges for $|x| > R$. The number R is called the radius of convergence.

Examples for (iii)
(a) $\sum x^n$ $R = 1$ the series diverges when $x = \pm 1$.
(b) $\sum \frac{x^n}{n^2}$ $R = 1$ the series converges when $x = \pm 1$.
(c) $\sum \frac{x^n}{n}$ $R = 1$ the series diverges when $x = 1$ and converges when $x = -1$.

These facts can be established using the following version of the ratio test

Ratio Test Suppose we have a series $\sum c_n$ of positive terms, and that the ratio $\frac{c_{n+1}}{c_n}$ tends to a limit L as $n \to \infty$. Then
(i) if $L < 1$ the series converges, (ii) if $L > 1$ the series diverges
(ii) if $L = 1$ the test gives no information

Examples for (iii)
(a) $\sum \frac{1}{n}$ $L = 1$ the series diverges. (b) $\sum \frac{1}{n^2}$ $L = 1$ the series converges.

Some revision notes on Geometric Series (GP)

Definition A finite GP is a series of the form

$$\sum_{k=0}^{n} ax^k = a + ax + ax^2 + \cdots + ax^n = a(1 + x + x^2 + \cdots + x^n).$$

The easiest way to verify the formula for the sum of such a GP (if you don't remember it) is to remember the trick of multiplying by *x*.

Let $S_n = a + ax + ax^2 + \cdots + ax^n$

$xS_n = \quad ax + ax^2 + \cdots + ax^n + ax^{n+1}.$

Subtracting then gives $(1-x)S_n = a(1-x^{n+1})$, so $S_n = a\dfrac{1-x^{n+1}}{1-x} \quad (x \neq 1)$.

Definition An infinite GP is a power series where all the coefficients are equal, i.e. a series of the form $\sum_{k=0}^{\infty} ax^n$.

Using the formula for the partial sum S_n derived above, we see that if $|x| < 1$ the series converges, with sum $\dfrac{a}{1-x}$. If $|x| > 1$ the series diverges. The same formula tells us that if $x = -1$ the series diverges. If $x = 1$ then $S_n = a(n+1)$, since all the terms of the series are equal to *a*, and so again the series diverges.

The trick of multiplying by *x* also works with some other series, for example

Let $T_n = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$

$xT_n = \quad x + 2x^2 + \cdots + (n-1)x^{n-1} + nx^n.$

Subtracting gives $(1-x)T_n = 1 + x + x^2 + \cdots + x^{n-1} - nx^n = \dfrac{1-x^n}{1-x} - nx^n$.

Therefore $T_n = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{1-x}$.

Details for Example 1

$f_n(x) = x^n$: $f_n(1) - f_n(x) = 1 - x^n > 0.999$ provided $x^n < 0.0001$,

which is satisfied provided $n \ln x < \ln(0.0001)$,

i.e., $n > \dfrac{\ln(0.0001)}{\ln x}$ (remembering that for numbers between 0 and 1 the logarithm is negative).

So for example if $x > 0.99999$ then $f_n(1) - f_n(x) > 0.999$ provided

$n > \dfrac{\ln(0.0001)}{\ln(0.99999)} = 921029.4\ldots$, i.e. provided $n > 921029$

In general if $x > 1 - \varepsilon$ then $f_n(1) - f_n(x) = 1 - x^n > 1 - \delta$ provided $n > \dfrac{\ln \delta}{\ln(1-\varepsilon)}$.

An example on changing the order of quantification

The following simple example shows that the order of quantification can make a difference to the meaning and the truth of a statement.

Consider the statement $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, y > x$. This says that given any real number x we can find a bigger one y. This is true, for example $y = x + 1$ will do.

Changing the order of quantification gives the statement $\exists y \in \mathbf{R}, \forall x \in \mathbf{R}, y > x$. This says that there is a real number y which is bigger than all real numbers, i.e. that there is a largest real number. This is false.

Detailed proof of Theorem 4

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$ denote the n-th partial sum of the series.

The sum of a finite number of continuous functions is a continuous function, so for all n, $S_n(x)$ is a continuous function.

$S_n(x) \to f(x)$ as $n \to \infty$ (this is the definition of the sum of an infinite series).

Uniform convergence of a series is defined as uniform convergence of the sequence of partial sums.

Theorem 1 therefore tells us that since $\left(S_n(x)\right)$ is a sequence of continuous functions converging uniformly to $f(x)$ that $f(x)$ is a continuous function.

Let $f_n(x) = x^n$. We show that $f_n(x) \to 0$ uniformly on the interval [0, 0.5].

Let $\varepsilon > 0$. Then $|x^n - 0| = |x^n| < (0.5)^n < \varepsilon$ provided $n > \dfrac{\ln \varepsilon}{\ln(0.5)}$. This is independent of x. hence the definition of uniform convergence is satisfied.

Geometrically, given the "ε-strip" $\{(x,y):0 \le x \le 0.5,\ 0 \le y \le \varepsilon\}$, the graphs of x^n all lie entirely within the strip for $n > \dfrac{\ln \varepsilon}{\ln(0.5)}$.

Proof of Theorem 5 from Theorem 2 and from the definition of uniform convergence

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Then $\left|S_n(x) - f(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \le \sum_{k=n+1}^{\infty} M_k$.

Therefore $\sup\left|S_n(x) - f(x)\right| \le \sum_{k=n+1}^{\infty} M_k$. Since $\sum_{k=1}^{\infty} M_k$ converges, $\sum_{k=n+1}^{\infty} M_k \to 0$ as $n \to \infty$.

Hence by Theorem 2 $S_n(x) \to f(x)$ uniformly.

Also, since $\sum_{k=1}^{\infty} M_k$ converges, given $\varepsilon > 0, \exists N, \forall n \ge N, \left|\sum_{k=n+1}^{\infty} M_k\right| < \varepsilon$. So we deduce that $\forall \varepsilon > 0, \exists N, \forall n \ge N, \left|S_n(x) - f(x)\right| < \varepsilon$. So the definition of uniform convergence is satisfied.

Details of the argument in Theorem 7

We start from the statement

$$\forall\varepsilon > 0, \exists N, \forall n \geq N, S(\alpha, f_n) - \varepsilon \leq S(\alpha, f) \leq S(\alpha, f_n) + \varepsilon. \qquad \text{(A)}$$

This is true for all subdivisions α.

We also have, for all subdivisions α, $S(\alpha, f_n) \geq \inf\{S(\alpha, f_n)\} = \int_a^b f_n$.

So we deduce, using the left hand inequality from (A), that

$\int_a^b f_n - \varepsilon \leq S(\alpha, f)$ for all subdivisions α. The left hand side is therefore a lower bound for the set of all upper sums for f, and so $\int_a^b f_n - \varepsilon \leq \inf\{S(\alpha, f)\}$.

Now from the right hand inequality from (A) we deduce that $\inf\{S(\alpha, f)\} - \varepsilon \leq S(\alpha, f_n)$ for all subdivisions α.

The left hand side is therefore a lower bound for the set of all upper sums for f_n, and so $\inf\{S(\alpha, f)\} - \varepsilon \leq \inf\{S(\alpha, f_n)\} = \int_a^b f_n$. Hence $\inf\{S(\alpha, f)\} \leq \int_a^b f_n + \varepsilon$.

Proof that $\lim_{n\to\infty}\int_a^b f_n = \sup_\alpha\{s(\alpha, f)\}$.

$\forall\varepsilon > 0, \exists N, \forall n \geq N, \forall x \in [a,b], \left|f_n(x) - f(x)\right| < \frac{\varepsilon}{b-a}$,

i.e. $\forall\varepsilon > 0, \exists N, \forall n \geq N, \forall x \in [a,b],\ f_n(x) - \frac{\varepsilon}{b-a} < f(x) < f_n(x) + \frac{\varepsilon}{b-a}$.

Since the f_n, and therefore also f, are bounded on $[a,b]$, it follows that for any subset S of $[a,b]$,

$$\forall\varepsilon > 0, \exists N, \forall n \geq N, \inf_{x\in S}\left(f_n(x)\right) - \frac{\varepsilon}{b-a} \leq \inf_{x\in S}\left(f(x)\right) \leq \inf_{x\in S}\left(f_n(x)\right) + \frac{\varepsilon}{b-a}.$$

Now let α be any subdivision of $[a,b]$, and consider the lower sums. The inequality above tells us that $\forall\varepsilon > 0, \exists N, \forall n \geq N, m_i(f_n) - \frac{\varepsilon}{b-a} \leq m_i(f) \leq m_i(f_n) + \frac{\varepsilon}{b-a}$ for all i. We conclude that $\forall\varepsilon > 0, \exists N, \forall n \geq N, s(\alpha, f_n) - \varepsilon \leq s(\alpha, f) \leq s(\alpha, f_n) + \varepsilon$.

Now $\int_a^b f_n = \sup_\alpha\{s(\alpha, f_n)\}$, and so we conclude that

$$\forall\varepsilon > 0, \exists N, \forall n \geq N, \int_a^b f_n - \varepsilon \leq \sup_\alpha\{s(\alpha, f)\} \leq \int_a^b f_n + \varepsilon, \text{ i.e. } \lim_{n\to\infty}\int_a^b f_n = \sup_\alpha\{s(\alpha, f)\}.$$

Detailed proof of Theorem 8

$f(x) = \sum_{k=1}^{\infty} f_k(x)$. Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$.

A finite sum of Riemann integrable functions is Riemann integrable, so $S_n(x)$ in itegrable for all n.

Now $S_n(x) \to f(x)$ uniformly as $n \to \infty$, and so by Theorem 7

$$\int_a^b \sum_{k=1}^{\infty} f_k(x) = \int_a^b \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \int_a^b S_n(x) = \lim_{n\to\infty} \int_a^b \sum_{k=1}^{n} f_k(x)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \int_a^b f_k(x) = \sum_{k=1}^{\infty} \int_a^b f_k(x)$$

Series version of Theorem 9

Let $\sum_{k=1}^{\infty} f_k(x)$ be a series of differentiable functions, with continuous derivatives, converging pointwise to a function f on an open interval I (finite or infinite). Suppose that the series $\sum_{k=1}^{\infty} f_k'(x)$ of derivatives converges uniformly on I to a function g. Then f is differentiable and $f' = g$.

Proof

Let $S_n(x) = \sum_{k=1}^{n} f_k(x)$. Then the sequence $\left(S_n(x)\right)$ is a sequence of differentiable functions (a finite sum of differentiable functions is differentiable) converging pointwise on I to f.

The series $\sum_{k=1}^{\infty} f_k'(x)$ of derivatives converges uniformly on I, i.e. the sequence $S_n{}'(x) = \sum_{k=1}^{n} f_k'(x)$ converges uniformly on I to the function g.

Therefore by Theorem 9 f is differentiable and $f' = g$, i.e. $f'(x) = \sum_{k=1}^{\infty} f_k{}'(x)$.