FUNDAMENTALS OF ANALYSIS

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Chapter 8

UNIFORM CONVERGENCE

8.1. Introduction

We begin by making a somewhat familiar definition.

DEFINITION. Suppose that $f_n : X \to \mathbb{C}$ is a sequence of functions on a set $X \subseteq \mathbb{R}$. We say that the sequence f_n converges pointwise to the function $f : X \to \mathbb{C}$ if for every $x \in X$, we have

$$|f_n(x) - f(x)| \to 0 \qquad \text{as } n \to \infty.$$

EXAMPLE 8.1.1. Let $X = [0, 1]$. For every $n \in \mathbb{N}$ and every $x \in [0, 1]$, let $f_n(x) = x^n$. Then for every $x \in [0, 1]$, $f_n(x) \to f(x)$ as $n \to \infty$, where $f(x) = 0$ if $0 \le x < 1$ and $f(1) = 1$. Note that each of the functions $f_n(x)$ is continuous on $[0, 1]$, but the limit function $f(x)$ is not continuous on $[0, 1]$. Hence the continuity property of the functions $f_n(x)$ is not carried over to the limit function $f(x)$.

To carry over certain properties of the individual functions of a sequence to the limit function, we need a type of convergence which is stronger than pointwise convergence.

DEFINITION. Suppose that $f_n : X \to \mathbb{C}$ is a sequence of functions on a set $X \subseteq \mathbb{R}$. We say that the sequence f_n converges uniformly to the function $f : X \to \mathbb{C}$ if

$$\sup_{x \in X} |f_n(x) - f(x)| \to 0 \qquad \text{as } n \to \infty.$$

EXAMPLE 8.1.2. In Example 8.1.1, we have $f_n(x) \to f(x)$ pointwise in $[0, 1]$. However, if $0 \le x < 1$, then $|f_n(x) - f(x)| = x^n$ and so

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| \ge \sup_{x \in [0,1)} |f_n(x) - f(x)| = \sup_{x \in [0,1)} x^n = 1$$

for every $n \in \mathbb{N}$. It follows that $f_n(x) \to f(x)$ as $n \to \infty$, pointwise but not uniformly on $[0, 1]$.

REMARK. Pointwise convergence means that given any $\epsilon > 0$, for every $x \in X$, there exists $N = N(\epsilon, x)$ such that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{whenever } n > N(\epsilon, x).$$

Uniform convergence means that given any $\epsilon > 0$, there exists $N = N(\epsilon)$, independent of $x \in X$, such that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{whenever } n > N(\epsilon) \text{ and } x \in X.$$

8.2. Criteria for Uniform Convergence

We shall first of all extend the General principle of convergence to the case of uniform convergence.

THEOREM 8A. (GENERAL PRINCIPLE OF UNIFORM CONVERGENCE) *Suppose that f_n is a sequence of real or complex valued functions defined on a set $X \subseteq \mathbb{R}$. Then $f_n(x)$ converges uniformly on X as $n \to \infty$ if and only if, given any $\epsilon > 0$, there exists N such that*

$$\sup_{x \in X} |f_m(x) - f_n(x)| < \epsilon \qquad \textit{whenever } m > n \geq N.$$

PROOF. ($\Rightarrow$) Suppose that $f_n(x) \to f(x)$ uniformly on X as $n \to \infty$. Then given any $\epsilon > 0$, there exists N such that

$$\sup_{x \in X} |f_n(x) - f(x)| < \tfrac{1}{2}\epsilon \qquad \text{whenever } n \geq N.$$

It follows that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < \epsilon \qquad \text{whenever } m > n \geq N \text{ and } x \in X,$$

and so

$$\sup_{x \in X} |f_m(x) - f_n(x)| \leq \epsilon \qquad \text{whenever } m > n \geq N.$$

($\Leftarrow$) Since $\mathbb{R}$ and $\mathbb{C}$ are complete, for every $x \in X$, the sequence $f_n(x)$ converges pointwise to a limit $f(x)$, say, as $n \to \infty$. We shall show that $f_n(x) \to f(x)$ uniformly on X as $n \to \infty$. Given any $\epsilon > 0$, there exists N such that for every $x \in X$,

$$|f_m(x) - f_n(x)| < \epsilon \qquad \text{whenever } m > n \geq N.$$

Hence for every $x \in X$,

$$|f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f_n(x)| \leq \epsilon \qquad \text{whenever } n \geq N,$$

so that

$$\sup_{x \in X} |f_n(x) - f(x)| \leq \epsilon \qquad \text{whenever } n \geq N.$$

Hence $f_n(x) \to f(x)$ uniformly on X as $n \to \infty$. ♣

We next turn our attention to series of real or complex valued functions.

DEFINITION. Suppose that u_n is a sequence of real or complex valued functions defined on a set $X \subseteq \mathbb{R}$. We say that the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly on X if the sequence of partial sums

$$s_N(x) = \sum_{n=1}^{N} u_n(x)$$

converges uniformly on X.

We have the analogue of the Comparison test.

THEOREM 8B. (WEIERSTRASS'S M-TEST) *Suppose that u_n is a sequence of real or complex valued functions defined on a set $X \subseteq \mathbb{R}$. Suppose further that for every $n \in \mathbb{N}$, there exists a real constant M_n such that the series*

$$\sum_{n=1}^{\infty} M_n$$

is convergent, and that $|u_n(x)| \le M_n$ for every $x \in X$. Then the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly and absolutely on X.

PROOF. Given any $\epsilon > 0$, it follows from the General principle of convergence for series that there exists N such that

$$M_{n+1} + \ldots + M_n < \epsilon \qquad \text{whenever } m > n \ge N.$$

It follows that

$$|s_m(x) - s_n(x)| \le M_{n+1} + \ldots + M_n < \epsilon \qquad \text{whenever } m > n \ge N \text{ and } x \in X,$$

so that

$$\sup_{x \in X} |s_m(x) - s_n(x)| \le \epsilon \qquad \text{whenever } m > n \ge N.$$

It now follows from Theorem 8A that the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly on X. Note finally that absolute convergence follows pointwise from the proof of the Comparison test. ♣

The General principle of uniform convergence can also be used to establish the following two results.

THEOREM 8C. (DIRICHLET'S TEST) *Suppose that a_n and b_n are two sequences of real valued functions defined on a set $X \subseteq \mathbb{R}$, and satisfy the following conditions:*
(a) There exists $K \in \mathbb{R}$ such that $|s_n(x)| \le K$ for every $n \in \mathbb{N}$ and every $x \in X$, where $s_n(x)$ denotes the sequence of partial sums $s_n(x) = a_1(x) + \ldots + a_n(x)$.
(b) For every $x \in X$, the sequence $b_n(x)$ is monotonic.
(c) The sequence $b_n(x) \to 0$ uniformly on X as $n \to \infty$.

Then the series $\sum_{n=1}^{\infty} a_n(x)b_n(x)$ converges uniformly on X.

PROOF. Since $b_n(x) \to 0$ uniformly on X as $n \to \infty$, given any $\epsilon > 0$, there exists N_0 such that

$$|b_n(x)| < \frac{\epsilon}{4K} \qquad \text{whenever } n > N_0 \text{ and } x \in X.$$

It follows that whenever $M > N \geq N_0$, we have

$$\left|\sum_{n=N+1}^{M} a_n(x)b_n(x)\right| = |(s_{N+1}(x) - s_N(x))b_{N+1}(x) + \ldots + (s_M(x) - s_{M-1}(x))b_M(x)|$$
$$= |-s_N(x)b_{N+1}(x) + s_{N+1}(x)(b_{N+1}(x) - b_{N+2}(x)) + \ldots + s_{M-1}(x)(b_{M-1}(x) - b_M(x)) + s_M(x)b_M(x)|$$
$$\leq K(|b_{N+1}(x)| + |b_{N+1}(x) - b_{N+2}(x)| + \ldots + |b_{M-1}(x) - b_M(x)| + |b_M(x)|)$$
$$= K(|b_{N+1}(x)| + |b_{N+1}(x) - b_M(x)| + |b_M(x)|) \leq 2K(|b_{N+1}(x)| + |b_M(x)|) < \epsilon.$$

The result follows from the General principle of uniform convergence. ♣

THEOREM 8D. (ABEL'S TEST) *Suppose that a_n and b_n are two sequences of real valued functions defined on a set $X \subseteq \mathbb{R}$, and satisfy the following conditions:*

(a) The series $\sum_{n=1}^{\infty} a_n(x)$ converges uniformly on X.
(b) For every $x \in X$, the sequence $b_n(x)$ is monotonic.
(c) There exists $K \in \mathbb{R}$ such that $|b_n(x)| \leq K$ for every $n \in \mathbb{N}$ and every $x \in X$.

Then the series $\sum_{n=1}^{\infty} a_n(x)b_n(x)$ converges uniformly on X.

PROOF. Given any $\epsilon > 0$, there exists N_0 such that

$$\left|\sum_{n=N+1}^{m} a_n(x)\right| < \frac{\epsilon}{3K} \qquad \text{whenever } m > N \geq N_0 \text{ and } x \in X.$$

In other words, writing $s_n(x) = a_1(x) + \ldots + a_n(x)$, we have

$$|s_m(x) - s_N(x)| < \frac{\epsilon}{3K} \qquad \text{whenever } m > N \geq N_0 \text{ and } x \in X.$$

It follows that whenever $M > N \geq N_0$, we have

$$\begin{aligned}
\left|\sum_{m=N+1}^{M} a_m(x)b_m(x)\right| &= \left|\sum_{m=N+1}^{M} (s_m(x) - s_{m-1}(x))b_m(x)\right| \\
&= \left|\sum_{m=N+1}^{M} ((s_m(x) - s_N(x)) - (s_{m-1}(x) - s_N(x)))b_m(x)\right| \\
&= \left|\sum_{m=N+1}^{M} (s_m(x) - s_N(x))b_m(x) - \sum_{m=N+1}^{M-1} (s_m(x) - s_N(x))b_{m+1}(x)\right| \\
&\leq \sum_{m=N+1}^{M-1} |s_m(x) - s_N(x)||b_m(x) - b_{m+1}(x)| + |s_M(x) - s_N(x)||b_M(x)| \\
&< \frac{\epsilon}{3K} \sum_{m=N+1}^{M-1} |b_m(x) - b_{m+1}(x)| + \frac{\epsilon}{3K}|b_M(x)| \\
&= \frac{\epsilon}{3K} \left|\sum_{m=N+1}^{M-1} (b_m(x) - b_{m+1}(x))\right| + \frac{\epsilon}{3K}|b_M(x)| \\
&= \frac{\epsilon}{3K}|b_{N+1}(x) - b_M(x)| + \frac{\epsilon}{3K}|b_M(x)| \\
&\leq \frac{\epsilon}{3K}(|b_{N+1}(x)| + 2|b_M(x)|) \leq \epsilon.
\end{aligned}$$

The result follows from the General principle of uniform convergence. ♣

8.3. Consequences of Uniform Convergence

In this section, we discuss the implications of uniform convergence on continuity, integrability and differentiability. To answer the question first raised in Section 8.1, we have the following result.

THEOREM 8E. *Suppose that a sequence of functions $f_n : X \to \mathbb{C}$ converges uniformly on a set $X \subseteq \mathbb{R}$ to a function $f : X \to \mathbb{C}$ as $n \to \infty$. Suppose further that $c \in X$ and that the function f_n is continuous at c for every $n \in \mathbb{N}$. Then the function f is continuous at c.*

Remark. The conclusion of Theorem 8E can be written in the form

$$\lim_{x \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to c} f_n(x).$$

Theorem 8E then says that if the sequence of functions converges uniformly on X, then the order of the two limiting processes can be interchanged.

Proof of Theorem 8E. Given any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\sup_{x \in X} |f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

Since f_n is continuous at c, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(c)| < \frac{\epsilon}{3} \qquad \text{whenever } |x - c| < \delta.$$

It follows that whenever $|x - c| < \delta$, we have

$$|f(x) - f(c)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| < \epsilon.$$

Hence f is continuous at c. ♣

We immediately have the following corollary of Theorem 8E.

THEOREM 8F. *Suppose that u_n is a sequence of real or complex valued functions defined on a set $X \subseteq \mathbb{R}$, and that the series*

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly to a function $s(x)$ on X. Suppose further that $c \in X$ and that the function u_n is continuous at c for every $n \in \mathbb{N}$. Then the function s is continuous at c.

We next study the effect of uniform convergence on integrability.

THEOREM 8G. *Suppose that f_n is a sequence of real valued functions integrable over a closed interval $[A, B]$. Suppose further that $f_n \to f$ uniformly on $[A, B]$ as $n \to \infty$. Then the function f is integrable over $[A, B]$, and*

$$\int_A^B f(x)\,\mathrm{d}x = \lim_{n \to \infty} \int_A^B f_n(x)\,\mathrm{d}x. \tag{1}$$

Remark. The conclusion of Theorem 8G can be written in the form

$$\int_A^B \left(\lim_{n \to \infty} f_n(x) \right) \mathrm{d}x = \lim_{n \to \infty} \int_A^B f_n(x)\,\mathrm{d}x.$$

Theorem 8G then says that if the sequence of functions converges uniformly on $[A, B]$, then the order of integration and taking limits as $n \to \infty$ can be interchanged.

Proof of Theorem 8G. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{x\in[A,B]} |f_n(x) - f(x)| < \frac{\epsilon}{3(B-A)} \qquad \text{whenever } n \geq N. \tag{2}$$

It follows in particular that

$$f_N(x) - \frac{\epsilon}{3(B-A)} < f(x) < f_N(x) + \frac{\epsilon}{3(B-A)} \qquad \text{whenever } x \in [A,B].$$

Hence for any dissection Δ of $[A,B]$, we have

$$s(f_N,\Delta) - \frac{\epsilon}{3} \leq s(f,\Delta) \leq S(f,\Delta) \leq S(f_N,\Delta) + \frac{\epsilon}{3},$$

so that

$$S(f,\Delta) - s(f,\Delta) \leq S(f_N,\Delta) - s(f_N,\Delta) + \frac{2\epsilon}{3}.$$

Since f_N is integrable over $[A,B]$, there exists a dissection Δ of $[A,B]$ such that

$$S(f_N,\Delta) - s(f_N,\Delta) < \frac{\epsilon}{3}, \qquad \text{so that} \qquad S(f,\Delta) - s(f,\Delta) < \epsilon.$$

Hence f is integrable over $[A,B]$. On the other hand, it follows from (2) that

$$\left| \int_A^B f_n(x)\,\mathrm{d}x - \int_A^B f(x)\,\mathrm{d}x \right| \leq \int_A^B |f_n(x) - f(x)|\,\mathrm{d}x < \epsilon \qquad \text{whenever } n \geq N.$$

The assertion (1) follows immediately. ♣

We immediately have the following corollary of Theorem 8G.

THEOREM 8H. *Suppose that u_n is a sequence of real valued functions defined on a closed interval $[A,B]$, and that the series*

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly to a function $s(x)$ on $[A,B]$. Suppose further that the function u_n is integrable over $[A,B]$ for every $n \in \mathbb{N}$. Then the function s is integrable over $[A,B]$, and

$$\int_A^B s(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_A^B u_n(x)\,\mathrm{d}x.$$

Remark. The conclusion of Theorem 8H can be written in the form

$$\int_A^B \left(\sum_{n=1}^{\infty} u_n(x) \right) \mathrm{d}x = \sum_{n=1}^{\infty} \int_A^B u_n(x)\,\mathrm{d}x.$$

Theorem 8H then says that if the sequence of functions converges uniformly on $[A,B]$, then the order of integration and summation can be interchanged. In other words, the series can be integrated term by term.

We next study the effect of uniform convergence on differentiability.

THEOREM 8J. *Suppose that f_n is a sequence of real valued functions differentiable in a closed interval $[A,B]$; in other words, differentiable at every point in the open interval (A,B), right differentiable at A and left differentiable at B. Suppose further that the sequence $f_n(x_0)$ converges for some $x_0 \in [A,B]$, and that the sequence f_n' converges uniformly on $[A,B]$. Then the sequence f_n converges uniformly on $[A,B]$, and the limit function f is differentiable in $[A,B]$. Furthermore, for every $x \in [A,B]$, we have*

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Remark. The conclusion of Theorem 8J can be written in the form

$$\left(\lim_{n\to\infty} f_n(x)\right)' = \lim_{n\to\infty} f_n'(x).$$

Theorem 8J then says essentially that if the sequence of functions satisfies some mild convergence property and the sequence of derivatives converges uniformly on $[A,B]$, then the order of differentiation and taking limits as $n \to \infty$ can be interchanged.

Proof of Theorem 8J. Suppose that $f_n' \to g$ as $n \to \infty$. Since the convergence is uniform in $[A,B]$, given any $\epsilon > 0$, there exists N such that

$$\sup_{[A,B]} |f_n'(x) - g(x)| < \frac{\epsilon}{4(1+(B-A))} \qquad \text{whenever } n \geq N, \tag{3}$$

so that

$$\sup_{[A,B]} |f_m'(x) - f_n'(x)| < \frac{\epsilon}{2(1+(B-A))} \qquad \text{whenever } m > n \geq N.$$

Suppose that $\eta_1, \eta_2 \in [A,B]$. Applying the Mean value theorem to the function $f_m - f_n$, we have

$$\begin{aligned} |(f_m(\eta_1) - f_n(\eta_1)) - (f_m(\eta_2) - f_n(\eta_2))| &= |\eta_1 - \eta_2||f_m'(\xi) - f_n'(\xi)| \\ &< |\eta_1 - \eta_2| \frac{\epsilon}{2(1+(B-A))} < \frac{\epsilon}{2} \end{aligned} \tag{4}$$

for some ξ between η_1 and η_2. On the other hand, since $f_n(x_0)$ converges as $n \to \infty$, there exists N' such that

$$|f_m(x_0) - f_n(x_0)| < \frac{\epsilon}{4} \qquad \text{whenever } m > n \geq N'.$$

It follows from (4), with $\eta_1 = x$ and $\eta_2 = x_0$, that

$$|f_m(x) - f_n(x)| < |f_m(x_0) - f_n(x_0)| + \frac{\epsilon}{2} < \frac{3\epsilon}{4} \qquad \text{whenever } m > n \geq \max\{N, N'\},$$

and so it follows from the Principle of uniform convergence that $f_n(x)$ converges uniformly in $[A,B]$. Suppose that $f_n(x) \to f(x)$ as $n \to \infty$. Let $c \in [A,B]$ be fixed. For every $x \in [A,B]$, it follows from (4), with $\eta_1 = x$ and $\eta_2 = c$, that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| < \frac{\epsilon}{2} \qquad \text{whenever } m > n \geq N,$$

so that on letting $m \to \infty$, we have

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| < \frac{\epsilon}{2}. \tag{5}$$

Since f_N is differentiable at c, there exists $\delta > 0$ such that

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f_N'(c) \right| < \frac{\epsilon}{4} \qquad \text{whenever } 0 < |x - c| < \delta \text{ and } x \in [A,B]. \tag{6}$$

Combining (5), (6) and (3), we conclude that

$$\left|\frac{f(x)-f(c)}{x-c}-g(c)\right| \le \left|\frac{f(x)-f(c)}{x-c}-\frac{f_N(x)-f_N(c)}{x-c}\right| + \left|\frac{f_N(x)-f_N(c)}{x-c}-f_N'(c)\right| + |f_N'(c)-g(c)| < \epsilon$$

whenever $0<|x-c|<\delta$ and $x\in[A,B]$. Hence

$$f'(c)=g(c)=\lim_{n\to\infty} f_n'(c). \qquad \clubsuit$$

We immediately have the following corollary of Theorem 8J.

THEOREM 8K. *Suppose that u_n is a sequence of real valued functions differentiable in a closed interval $[A,B]$. Suppose further that the series*

$$\sum_{n=1}^{\infty} u_n(x_0)$$

converges for some $x_0\in[A,B]$, and that the series

$$\sum_{n=1}^{\infty} u_n'(x)$$

converges uniformly on $[A,B]$. Then the series

$$\sum_{n=1}^{\infty} u_n(x)$$

converges uniformly on $[A,B]$, and its sum $s(x)$ is differentiable in $[A,B]$. Furthermore, for every $x\in[A,B]$, we have

$$s'(x)=\sum_{n=1}^{\infty} u_n'(x).$$

Remark. The conclusion of Theorem 8K can be written in the form

$$\left(\sum_{n=1}^{\infty} u_n(x)\right)' = \sum_{n=1}^{\infty} u_n'(x).$$

Theorem 8K then says essentially that if the series of functions satisfies some mild convergence property and the series of derivatives converges uniformly on $[A,B]$, then the order of differentiation and summation can be interchanged.

8.4. Application to Power Series

Consider a power series in $z\in\mathbb{C}$, of the form

$$\sum_{n=0}^{\infty} a_n z^n, \tag{7}$$

where $a_n\in\mathbb{C}$ for every $n\in\mathbb{N}\cup\{0\}$. Recall Theorem 3Q, that if the power series (7) has radius of convergence R and if $0<r<R$, then the series

$$\sum_{n=0}^{\infty} |a_n| r^n$$

converges. It follows from Weierstrass's M-test that the power series (7) converges uniformly on the set $\{z \in \mathbb{C} : |z| \leq r\}$. Suppose now that $|z_0| < R$. Then there exists r such that $|z_0| < r < R$. It follows from Theorem 8F that the power series is continuous at z_0. We have therefore proved the following result.

THEOREM 8L. *Suppose that the power series (7) has radius of convergence R. Then for every r satisfying $0 < r < R$, the power series converges uniformly on the set $\{z \in \mathbb{C} : |z| \leq r\}$. Furthermore, the sum of the power series is continuous on the set $\{z \in \mathbb{C} : |z| < R\}$.*

We next consider real power series.

THEOREM 8M. *Suppose that the real power series*

$$\sum_{n=0}^{\infty} a_n x^n, \tag{8}$$

where $a_n \in \mathbb{R}$ for every $n \in \mathbb{N} \cup \{0\}$, converges in the interval $(-R, R)$ to a function $f(x)$. Then $f(x)$ is differentiable on $(-R, R)$, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

On the other hand, if $|X| < R$, then

$$\int_0^X f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1} X^{n+1}.$$

PROOF. It is not difficult to see that the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \tag{9}$$

converges in the interval $(-R, R)$. It follows from Theorem 8L that the series (9) converges uniformly on any closed subinterval of $(-R, R)$. The first assertion follows from Theorem 8K. The second assertion follows from Theorem 8H on noting that the power series converges uniformly on the closed interval with endpoints 0 and X. ♣

We conclude this chapter by establishing the following useful result.

THEOREM 8N. (ABEL'S THEOREM) *Suppose that the real series*

$$\sum_{n=0}^{\infty} a_n$$

is convergent. Then

$$\sum_{n=0}^{\infty} a_n x^n \to \sum_{n=0}^{\infty} a_n \qquad \text{as } x \to 1- .$$

PROOF. It follows from Abel's test that the series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges uniformly on $[0, 1]$. Let $s(x)$ be its sum. Then it follows from Theorem 8F that $s(x)$ is continuous on $[0, 1]$. In particular, we have $s(x) \to s(1)$ as $x \to 1-$. ♣

Problems for Chapter 8

1. For each of the following, prove that the sequence of functions converges pointwise on its domain of definition as $n \to \infty$, and determine whether the convergence is uniform on this set:
   a) $f_n(x) = \dfrac{nx}{n+x}$ on $[0, \infty)$
   b) $f_n(x) = \dfrac{nx}{1+n^2x^2}$ on $[0, \infty)$
   c) $f_n(x) = x^n(1-x)$ on $[0, 1]$
   d) $f_n(x) = \dfrac{\sin nx}{nx}$ on $(0, 1)$
   e) $f_n(x) = nx\mathrm{e}^{-nx^2}$ on $[0, 1]$

2. Suppose that f_n and g_n are complex valued functions defined on a set $X \subseteq \mathbb{R}$. Suppose further that $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ as $n \to \infty$ uniformly on X.
   a) Prove that $\alpha f_n(x) + \beta g_n(x) \to \alpha f(x) + \beta g(x)$ as $n \to \infty$ uniformly on X for any $\alpha, \beta \in \mathbb{C}$.
   b) Is it necessarily true that $f_n(x)g_n(x) \to f(x)g(x)$ as $n \to \infty$ uniformly on X? Justify your assertion.

3. a) Suppose that $f_n(x) \to f(x)$ as $n \to \infty$ uniformly on each of the sets $X_1, \ldots, X_k$ in $\mathbb{R}$. Prove that $f_n(x) \to f(x)$ as $n \to \infty$ uniformly on the union $X_1 \cup \ldots \cup X_k$.
   b) Give an example to show that the analogue for an infinite collection of sets does not hold.

4. The series $\displaystyle\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent on a set $S \subseteq \mathbb{R}$.
   a) Is the series necessarily absolutely convergent for every $x \in S$? Justify your assertion.
   b) Is the series necessarily absolutely convergent for some $x \in S$? Justify your assertion.

5. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(1+x^{2n})}$ converges uniformly on $\mathbb{R}$.

6. Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent real series.
   a) Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$.
   b) Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n n^{-x}$ converges uniformly on $[0, \infty)$.

7. For every $n \in \mathbb{N}$, let $f_n(x) = n^{-1}\mathrm{e}^{-x/n}$.
   a) Show that $f_n(x)$ converges uniformly on $(0, \infty)$.
   b) Show that $\displaystyle\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,\mathrm{d}x$ and $\displaystyle\int_0^{\infty} \Big(\lim_{n\to\infty} f_n(x)\Big)\,\mathrm{d}x$ both exist but are not equal.

8. For every $n \in \mathbb{N}$, let $f_n(x) = \dfrac{x^n}{1+x^{2n}}$.
   a) For what values of $x \in \mathbb{R}$ does $f_n(x)$ converge? Find the limit function $f(x)$ for these values.
   b) Prove that $f_n(x)$ converges uniformly on any interval $[A, B]$ in $\mathbb{R}$ such that
      (i) $[A, B] \subseteq (-\infty, -1)$; or
      (ii) $[A, B] \subseteq (-1, 1)$; or
      (iii) $[A, B] \subseteq (1, \infty)$.
   c) Can $f_n(x)$ converge uniformly on any interval $I \subseteq \mathbb{R}$ such that $1 \in I$? Justify your assertion.

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