

**Faculty of Mathematical Studies**  
**MA204 Real Analysis: Riemann Integration**

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## Introduction: Aims

Ideas about integration have been around much longer than those of differentiation. The Greek mathematician Archimedes (3rd Century B.C.) knew how to calculate the area of a segment of a parabola by “quadrature”, which involved approximation by regions of known area in much the same way that numerical integration routines like the trapezium rule do. Integration as the reverse of differentiation was a much later idea, after the invention of the differential calculus in the time of Newton (17th Century A.D.). Bringing the two views of integration together was the work of mathematicians in the 19th Century in particular, culminating in the work of Bernhard Riemann (1826-1866), whose name is associated with the theory of integration we shall develop in this section of the course.

The aim of this section of the course is to study the Riemann Theory of integration, to understand the use of the completeness of the real numbers in formulating the theory, and to establish the connection with anti-differentiation through the Fundamental Theorem

## Prerequisites: Revision

MA201, in particular the section on Real Numbers. Least upper bounds and greatest lower bounds, their definitions and properties

**Definition 1** A set  $S$  of real numbers is said to be *bounded above* if

$$\exists K \in \mathbf{R}, \forall x \in S, x \leq K.$$

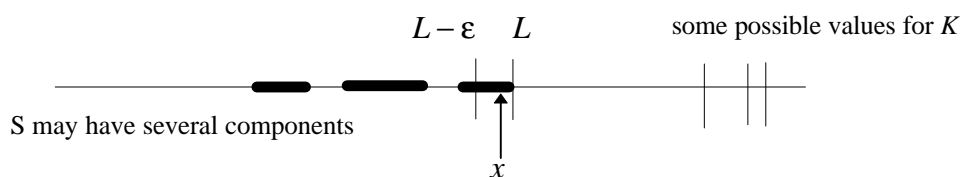
Any such  $K$  is said to be *an upper bound* for  $S$ .

**Definition 2** Given a set  $S$ , bounded above, a number  $L$  is said to be a *least upper bound*, or *supremum*, of  $S$  if

- (a)  $L$  is an upper bound for  $S$ , and
- (b)  $\forall \varepsilon > 0, \exists x \in S, x > L - \varepsilon$ .

It can be shown that when a set has a supremum, it is unique. We write  $L = \sup(S)$ .

On the number line we have the following picture:



**Exercise 1** Write down analogous definitions for a set being *bounded below*, and for the *greatest lower bound* (or *infimum*). We write  $G = \inf(S)$ . Draw an analogous picture on the real number line.

Click [here](#) to see a solution

## Upper and lower Riemann sums

The basic idea behind the Riemann integral is that of the area under a curve. We approximate that area by rectangles, from below and from above, and then say that the function is Riemann integrable if the approximations can be made as close as we specify. In order to formulate this idea on a properly analytic basis we need to introduce some notation and terminology.

Throughout this development we shall restrict ourselves to bounded functions defined on a finite interval  $[a,b]$ . We let  $m$  and  $M$  denote the greatest lower bound and the least upper bound respectively of the function  $f$  over the interval  $[a,b]$ . Symbolically

$$m = \inf\{f(x):a \leq x \leq b\} \text{ and } M = \sup\{f(x):a \leq x \leq b\}.$$

We now need to subdivide the interval  $[a,b]$  into small sub-intervals, which will serve as bases for the rectangles used to approximate to the area. We shall use Greek letters to denote subdivisions, so

$$\alpha = (x_0, x_1, x_2, \dots, x_{n-1}, x_n), \text{ where } x_0 = a \text{ and } x_n = b.$$

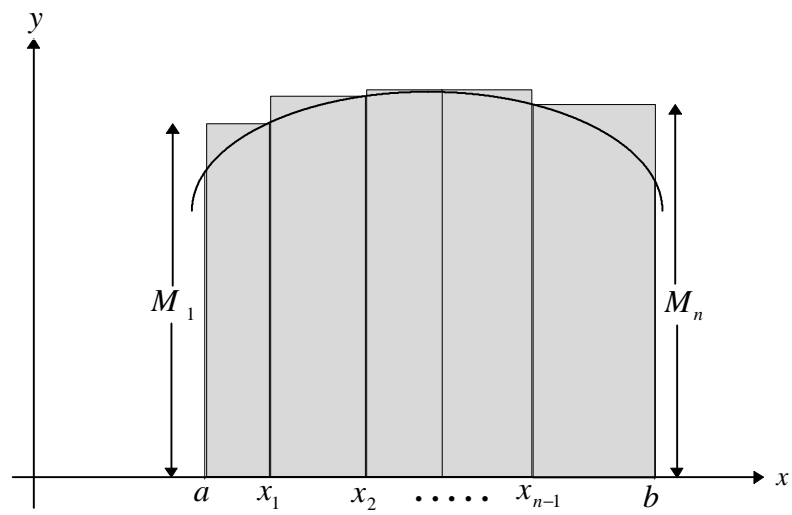
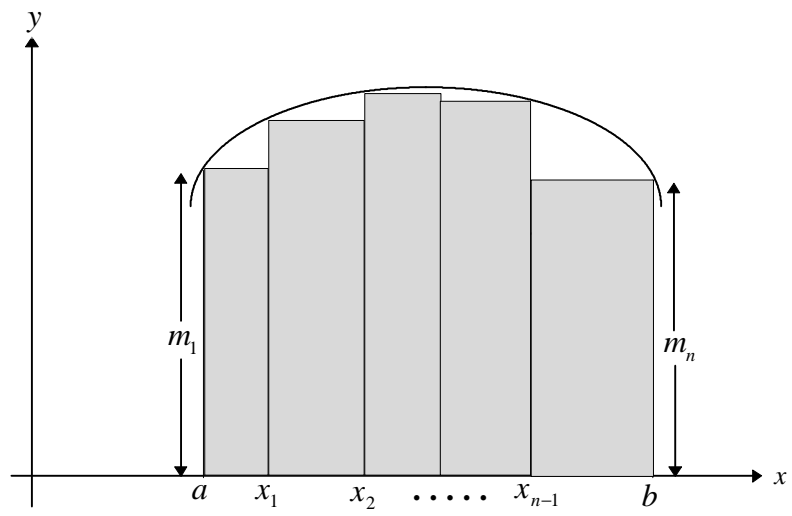
We then need to consider the greatest lower bound and least upper bound of the function over each of the sub-intervals. So we define

$$m_i = \inf\{f(x):x_{i-1} \leq x \leq x_i\} \text{ and } M_i = \sup\{f(x):x_{i-1} \leq x \leq x_i\} \text{ for } 1 \leq i \leq n.$$

These quantities are illustrated on the diagrams on the next page

The sum of the areas of the rectangles lying below the graph is called the lower sum corresponding to the subdivision, and the sum of the rectangles enclosing the area underneath the graph is called the upper sum corresponding to the subdivision. We introduce the notation for these as follows

$$s(\alpha) = \sum_{i=1}^n m_i(x_i - x_{i-1}); \quad S(\alpha) = \sum_{i=1}^n M_i(x_i - x_{i-1}).$$



## Upper and lower Riemann integral: the Riemann integral defined

The idea now is to see how large the lower sums can become. In analytical terms we look for the least upper bound of the set of lower sums. To know that this exists we must first show that this set is bounded above.

**Theorem 1** For all subdivisions  $\alpha$ , we have

$$m(b-a) \leq s(\alpha) \leq M(b-a) \text{ and } m(b-a) \leq S(\alpha) \leq M(b-a).$$

**Proof** For  $i = 1, \dots, n$ ,  $m \leq m_i \leq M$ , and so

$$\sum_{i=1}^n m(x_i - x_{i-1}) \leq \sum_{i=1}^n m_i(x_i - x_{i-1}) \leq \sum_{i=1}^n M(x_i - x_{i-1}),$$

giving

$$m(b-a) \leq s(\alpha) \leq M(b-a).$$

**Exercise 2** Write out the version of this proof for upper sums, giving a full explanation for the last step.

Click [here](#) to see a solution

**Definition 3** We define the lower and upper Riemann integrals of  $f$  by

$$\int_a^b f = \sup_{\alpha} \{s(\alpha)\} \quad \int_a^b f = \inf_{\alpha} \{S(\alpha)\}.$$

**Theorem 2**  $m(b-a) \leq \int_a^b f \leq M(b-a)$  and  $m(b-a) \leq \int_a^b f \leq M(b-a)$ .

**Proof** The corresponding inequalities for all lower and upper sums are true by Theorem 1, and hence for the lower and upper integral respectively.

**Definition 4** The bounded function  $f$  is said to be integrable over  $[a,b]$  if the upper and lower integrals are equal. The common value is called the integral of  $f$  over  $[a,b]$ .

## Calculating Riemann sums: two examples

**Example 1** Let  $f(x) = x$ ;  $[a, b] = [1, 2]$ . Let  $\alpha$  be the subdivision

$$\left(1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \dots, 2\right) \text{ i.e., } x_i = 1 + \frac{i}{n}.$$

$$m_i = 1 + \frac{i-1}{n}; M_i = 1 + \frac{i}{n}; x_i - x_{i-1} = \frac{1}{n} \text{ for each } i.$$

$$s(\alpha) = \sum_{i=1}^n \left(1 + \frac{i-1}{n}\right) \frac{1}{n} = \sum_{i=1}^n \frac{1}{n} + \sum_{i=1}^n \frac{i-1}{n^2} = 1 + \frac{n-1}{2n} = \frac{3}{2} - \frac{1}{2n},$$

$$S(\alpha) = \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \frac{1}{n} = \sum_{i=1}^n \frac{1}{n} + \sum_{i=1}^n \frac{i}{n^2} = 1 + \frac{n+1}{2n} = \frac{3}{2} + \frac{1}{2n}.$$

**Example 2** Let  $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational,} \end{cases} \quad [a, b] = [0, 1].$

For any subdivision  $\alpha$  of  $[0, 1]$ ,  $m_i = 0$ ,  $M_i = 1$ , so  $s(\alpha) = 0$  and  $S(\alpha) = 1$ .

Therefore  $\int_a^b f = 0$  and  $\int_a^b f = 1$ . So  $f$  is not integrable over  $[0, 1]$ .

### Basic properties of upper and lower integrals

In order to show that the definition of the Riemann integral is a good one we need to establish that it has all the properties we associate with the anti-differentiation version of the integral. We shall also need to show that the Riemann integral for a function which has an anti-derivative is given the same value by both methods. This is what the Fundamental Theorem of Calculus says.

**Theorem 3** If  $\beta$  is a subdivision of  $[a,b]$  formed by adding a finite number of points to a subdivision  $\alpha$  of  $[a,b]$  then  $s(\alpha) \leq s(\beta)$  and  $S(\alpha) \geq S(\beta)$ .

**Proof** We prove the case of adding a single point  $c$ , where  $x_{i-1} < c < x_i$ .

$$m_i' = \inf\{f(x): x_{i-1} \leq x \leq c\},$$

$$m_i'' = \inf\{f(x): c \leq x \leq x_i\},$$

$$M_i' = \sup\{f(x): x_{i-1} \leq x \leq c\},$$

$$M_i'' = \sup\{f(x): c \leq x \leq x_i\}.$$

$$\text{Then } m_i \leq m_i', m_i \leq m_i'', M_i' \leq M_i, M_i'' \leq M_i.$$

We then have

$$m_i(x_i - x_{i-1}) = m_i(x_i - c) + m_i(c - x_{i-1}) \leq m_i''(x_i - c) + m_i'(c - x_{i-1}),$$

$$M_i(x_i - x_{i-1}) = M_i(x_i - c) + M_i(c - x_{i-1}) \geq M_i''(x_i - c) + M_i'(c - x_{i-1}).$$

Thus  $s(\alpha) \leq s(\beta)$  and  $S(\alpha) \geq S(\beta)$ .

The case for  $n$  points is proved by adding them one at a time (formally by induction).

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**Theorem 4** If  $a < b < c$  then  $\int_a^b f + \int_b^c f = \int_a^c f$ .

**Proof** Let  $\alpha_1, \alpha_2$  be two arbitrary subdivisions of  $[a, b]$  and  $[b, c]$  respectively.

Together they form a subdivision  $\alpha$  of  $[a, c]$ , and we have

$$s(\alpha_1) + s(\alpha_2) = s(\alpha) \leq \int_a^c f.$$

Since this is true for all  $\alpha_1, \alpha_2$ , we have

$$\int_a^b f + \int_b^c f \leq \int_a^c f.$$

Now let  $\beta$  be an arbitrary subdivision of  $[a, c]$ , and let  $\gamma$  be the subdivision formed by adding the point  $b$ . Then  $\gamma$  splits into two subdivisions  $\gamma_1, \gamma_2$  of  $[a, b]$  and  $[b, c]$  respectively, giving

$$s(\beta) \leq s(\gamma) = s(\gamma_1) + s(\gamma_2) \leq \int_a^b f + \int_b^c f.$$

Since this is true for all  $\beta$ , we have

$$\int_a^c f \leq \int_a^b f + \int_b^c f.$$

Hence equality.

**Exercise 3** Prove the analogous result for upper integrals.

Click [here](#) to see a solution

**Theorem 5** If  $a < b$  then  $\int_a^b f \leq S(\int_a^b f)$ .

**Proof** If  $\alpha$  is any subdivision of  $[a, b]$  then  $s(\alpha) \leq S(\alpha)$ . However this is not sufficient. We need to show that every lower sum is less than every upper sum, the two sums ranging independently over all possible subdivisions. So let  $\beta$  and  $\gamma$  be two arbitrary subdivisions of  $[a, b]$ . Now let  $\alpha$  be the subdivision formed by taking all the points of  $\beta$  and  $\gamma$  together. We then have, using **Theorem 3**,

$$s(\beta) \leq s(\alpha) \leq S(\alpha) \leq S(\gamma).$$

Hence the result.

## Integrability

**Definition 5**  $\int_a^a f = 0$ . If  $b < a$  then we define  $\int_a^b f = -\int_b^a f$ , with similar definitions for upper and lower integrals.

**Theorem 6** The bounded function  $f$  is integrable over the interval  $[a,b]$  if and only if, for every  $\varepsilon > 0$ , there is a subdivision  $\alpha$  of  $[a,b]$  for which  $S(\alpha) \leq s(\alpha) + \varepsilon$ .

**Proof** For every  $\varepsilon > 0$  there are subdivisions  $\beta$  and  $\gamma$  such that

$$s(\beta) > \int_a^b f - \varepsilon/2 \quad \text{and} \quad S(\gamma) < \int_a^b f + \varepsilon/2.$$

We now let  $\alpha$  be the subdivision obtained by including all the points from  $\beta$  and  $\gamma$ . If  $f$  is integrable then the upper and lower integrals are the same, so

$$S(\alpha) \leq S(\gamma) < \int_a^b f + \varepsilon/2 \leq s(\beta) + \varepsilon/2 + \varepsilon/2 \leq s(\alpha) + \varepsilon.$$

Conversely we have

$$\int_a^b f \leq S(\alpha) \leq s(\alpha) + \varepsilon \leq \int_a^b f + \varepsilon.$$

This implies that the upper integral is less than or equal to the lower integral. Theorem 5 gives us the reverse inequality. Therefore the upper and lower integrals are equal, and so  $f$  is integrable.

Referring back to **Example 1** we see that **Theorem 6** tells us that the function there is integrable. With **Example 2** on the other hand we see that the function in question is not integrable.

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**Theorem 7**  $\int_a^b f + \int_a^b g \leq \int_a^b (f + g)$  and  $\int_a^b f + \int_a^b g \geq \int_a^b (f + g)$ .

**Proof** For any interval in a subdivision we have

$$m_i(f) + m_i(g) \leq m_i(f + g). \quad \text{click [here](#) for details}$$

So for any subdivision  $\alpha$  we have

$$s(\alpha, f) + s(\alpha, g) \leq s(\alpha, f + g).$$

We cannot simply take the supremum of each term on the left hand side, since  $\sup(A) + \sup(B)$  can sometimes be greater than  $\sup(A \oplus B)$ .

The argument is more subtle, and uses property (b) of the supremum, as follows.

For any  $\varepsilon > 0$  there exist subdivisions  $\beta$  and  $\gamma$  such that

$$s(\beta, f) > \int_a^b f - \varepsilon/2 \quad \text{and} \quad s(\gamma, g) > \int_a^b g - \varepsilon/2.$$

We now let  $\alpha$  be the subdivision obtained by including all the points from  $\beta$  and  $\gamma$ , so

$$s(\alpha, f) \geq s(\beta, f) \quad \text{and} \quad s(\alpha, g) \geq s(\gamma, g).$$

Combining the three sets of inequalities above gives

$$\int_a^b f + \int_a^b g - \varepsilon < s(\alpha, f) + s(\alpha, g) \leq s(\alpha, f + g) \leq \int_a^b (f + g).$$

This is true for all  $\varepsilon > 0$  and so

$$\int_a^b f + \int_a^b g \leq \int_a^b (f + g).$$

**Exercise 4** Prove the result for upper integrals.

Click [here](#) to see a solution

**Corollary** If  $f$  and  $g$  are integrable, so is  $f + g$ , and  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ .

## Classes of integrable functions: continuous and monotonic

So far we have come across conditions for integrability, but have not established any classes of integrable functions. In the next section we consider the lower integral as a function of its upper limit and use the results to prove that continuous functions are integrable. In the next two theorems we shall use the notation

$$F(x) = \int_a^x f(t) dt.$$

**Theorem 8**  $F(x)$  is continuous in  $[a, b]$ .

**Proof** Let  $c$  and  $c+h$  both be numbers in  $[a, b]$ , where  $h > 0$ . We then have

$$F(c+h) - F(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f.$$

Using **Theorem 2** enables us to deduce that  $mh \leq F(c+h) - F(c) \leq Mh$ , using the notation introduced at the beginning of this topic, where we restricted discussion to the integration of bounded functions. It follows that  $\lim_{h \rightarrow 0^+} F(c+h) - F(c) = 0$ .

**Exercise 5** Prove the analogous result for the case  $h < 0$ . Both results together show that  $F$  is continuous.

Click [here](#) to see a solution

**Theorem 9** If  $f$  is continuous at  $c \in (a, b)$  then  $F$  is differentiable at  $c$ , and  $F'(c) = f(c)$ .

**Proof** Let  $\varepsilon$  be an arbitrary positive number. Because  $f$  is continuous at  $c$  there is a number  $\delta > 0$  such that for  $c - \delta \leq t \leq c + \delta$ ,  $f(c) - \varepsilon < f(t) < f(c) + \varepsilon$ .

Using these bounds and the result of **Theorem 2** tells us that for  $0 < h < \delta$ ,

$$(f(c) - \varepsilon)h < F(c+h) - F(c) < (f(c) + \varepsilon)h, \quad \text{i.e.} \quad \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon.$$

Hence the result. As in Theorem 8 the case where  $h$  is negative is left as an exercise.

**Exercise 6** Prove the results of the last two theorems for the upper integral.

Click [here](#) to see a solution

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**Theorem 10** If  $f$  is bounded in  $[a,b]$  and continuous in  $(a,b)$  then  $f$  is integrable.

**Proof** Let  $G(x) = \int_a^x f - \int_a^x f$ .  $G$  is continuous in  $[a,b]$  and differentiable in  $(a,b)$ . Hence, by the Mean Value Theorem, there is a number  $c \in (a,b)$  for which  $G(b) - G(a) = G'(c)(b-a)$ . But  $G'(c) = 0$  and  $G(a) = 0$ . Thus  $G(b) = 0$  and so  $\int_a^b f - \int_a^b f = 0$ , i.e.  $f$  is integrable.

**Theorem 11** If  $f$  has a finite number of discontinuities then it is integrable.

**Proof** (outline) We need the result that if  $f$  is integrable over  $[a,b]$  and over  $[b,c]$  then  $f$  is integrable over  $[a,c]$ . We then split the interval at the discontinuities and use the last theorem in each separate interval.

Click [here](#) for a detailed proof

**Theorem 12** If  $f$  is increasing on  $[a,b]$  then  $f$  is integrable.

**Proof** Let  $\alpha$  be the subdivision obtained by dividing  $[a,b]$  into  $n$  equal subintervals.

So for  $i = 1, 2, \dots, n$ ,  $(x_i - x_{i-1}) = (b-a)/n$ .

Since  $f$  is increasing,  $m_i = f(x_{i-1})$ ;  $M_i = f(x_i)$ .

Calculating the upper and lower sums gives

$$\begin{aligned} S(\alpha) - s(\alpha) &= \sum_{i=1}^n M_i \left( \frac{b-a}{n} \right) - \sum_{i=1}^n m_i \left( \frac{b-a}{n} \right) \\ &= \frac{b-a}{n} \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \\ &= \frac{b-a}{n} (f(b) - f(a)) < \varepsilon \end{aligned}$$

provided  $n$  is sufficiently large. Hence the result, by **Theorem 6**.

**Note** This result is true irrespective of the discontinuities of  $f$ .

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**Theorem 13** If  $f$  and  $g$  are integrable so is  $fg$ .

**Proof** We first prove that  $f^2$  is integrable. Since  $f$  is bounded

$$\exists M, \forall x \in [a, b], |f(x)| \leq M.$$

$$|f^2(x) - f^2(x')| = |f(x) + f(x')||f(x) - f(x')| \leq 2M|f(x) - f(x')|,$$

$$\text{so for } i = 1, 2, \dots, n, M_i(f^2) - m_i(f^2) \leq 2M(M_i(f) - m_i(f)).$$

$$\text{Thus } S(\alpha, f^2) - s(\alpha, f^2) \leq 2M(S(\alpha, f) - s(\alpha, f)).$$

Since the function  $f$  is integrable we can choose the subdivision  $\alpha$  so that  $S(\alpha, f) - s(\alpha, f) < \varepsilon/2M$ . So  $f^2$  is integrable.

To prove that  $fg$  is integrable we use the identity  $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$ .

**Theorem 14** If  $f$  is integrable, so is  $|f|$ , and  $\left| \int_a^b f \right| \leq \int_a^b |f|$ .

**Proof**  $\forall x, x' \in (x_{i-1}, x_i), |f(x)| - |f(x')| \leq |f(x) - f(x')| \leq M_i(f) - m_i(f)$ .

$$\text{So } M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f),$$

$$\text{and hence } S(\alpha, |f|) - s(\alpha, |f|) \leq S(\alpha, f) - s(\alpha, f).$$

Now because  $f$  is integrable we can choose  $\alpha$  so that  $S(\alpha, f) - s(\alpha, f) < \varepsilon$ , and the result follows.

Finally  $-|f(x)| \leq f(x) \leq |f(x)|$  for all  $x$ , so  $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$ , i.e.  $\left| \int_a^b f \right| \leq \int_a^b |f|$ .

## Integrability using sequences of subdivisions I

**Theorem 15** Let  $f$  be a function bounded on  $[a, b]$ . Suppose that there is a sequence of subdivisions  $(\alpha_n)$  of  $[a, b]$  and a number  $K$  such that

$$S(\alpha_n) \rightarrow K \text{ and } s(\alpha_n) \rightarrow K \text{ as } n \rightarrow \infty.$$

Then  $f$  is integrable and  $\int_a^b f = K$ .

### Proof

(a) We first give a proof that  $f$  is integrable, as an illustration of the use of **Theorem 6**. This proof will in fact be superseded by (b) below, which establishes integrability and the value of the integral together.

We note that the conditions on the two sums imply that  $S(\alpha_n) - s(\alpha_n) \rightarrow 0$ , i.e.

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, S(\alpha_n) - s(\alpha_n) < \varepsilon.$$

**Theorem 6** then implies integrability.

(b) We use part of the limit definitions applied to the upper and lower sums.

$$\forall \varepsilon > 0, \exists M, \forall n \geq M, S(\alpha_n) < K + \varepsilon.$$

Hence  $\inf(S(\alpha)) \leq K$ . Also

$$\forall \varepsilon > 0, \exists T, \forall n \geq T, s(\alpha_n) > K - \varepsilon.$$

Hence  $\sup(s(\alpha)) \geq K$ . Using these inequalities we have

$$K \leq \sup(s(\alpha)) = \int_a^b f \leq \int_a^b f = \inf(S(\alpha)) \leq K.$$

So we have equality throughout, proving the result.

An example of the application of this theorem is on the next page

**Example 3** Let  $f(x) = x^3$  for  $x \in [1,2]$ . We consider the subdivision of this interval  $\alpha_n = (1, x, x^2, \dots, x^n = 2)$ , i.e. where  $x = \sqrt[n]{2}$ . We then have

$$m_i = (x^{i-1})^3 \text{ and } M_i = (x^i)^3, \text{ so}$$

$$s(\alpha_n) = \sum_{i=1}^n (x^{i-1})^3 (x^i - x^{i-1});$$

$$S(\alpha_n) = \sum_{i=1}^n (x^i)^3 (x^i - x^{i-1}).$$

We now evaluate the lower sum, which involves GPs.

$$\begin{aligned} s(\alpha_n) &= \sum_{i=1}^n x^{4i-3} - \sum_{i=1}^n x^{4i-4} = x \left( 1 + x^4 + x^8 + \dots + (x^4)^{n-1} \right) - \left( 1 + x^4 + x^8 + \dots + (x^4)^{n-1} \right) \\ &= (x-1) \left( \frac{x^{4n} - 1}{x^4 - 1} \right) = \left( \frac{x^{4n} - 1}{(x+1)(x^2+1)} \right) = \frac{16-1}{(\sqrt[n]{2}+1)(\sqrt[n]{4}+1)} \rightarrow \frac{15}{4} \text{ as } n \rightarrow \infty. \end{aligned}$$

A similar calculation, left to the reader, shows that  $S(\alpha_n)$  has the same limit. Hence  $x^3$  is integrable over  $[1,2]$  with value  $15/4$ . Click [here](#) for the details

## Integrability using sequences of subdivisions II

**Theorem 16** Let  $f$  be a function bounded on  $[a,b]$ , whose upper integral is  $K$ . Suppose that  $(\alpha_n)$  is a sequence of subdivisions of  $[a,b]$  having the property that the length of the largest subinterval of  $\alpha_n$  tends to zero as  $n \rightarrow \infty$ .

Then  $S(\alpha_n) \rightarrow K$  as  $n \rightarrow \infty$ .

**Proof** Note: This is probably the most analytically complex proof in this section of the course.

Given  $\varepsilon > 0$ , there is a subdivision  $\beta$  of  $[a,b]$  satisfying  $S(\beta) < K + \varepsilon/2$ . Let  $p$  be the number of sub-intervals (so that  $\beta$  consists of the points  $a, b$  and  $p - 1$  other points).

Given  $\eta > 0$ , there is an integer  $N$  such that for all  $n \geq N$  the length of the largest sub-interval of  $\alpha_n$  is less than  $\eta$ .

Now let  $\gamma$  be the subdivision obtained by adding the points of  $\beta$  to those of  $\alpha_n$ .

We can write  $\alpha_n = (a = x_0, x_1, x_2, \dots, x_r = b)$ .

Using the notation of Theorem 3, if a point  $c$  from  $\beta$  is added in the interval  $(x_{i-1}, x_i)$  then a contribution of  $M_i(x_i - x_{i-1})$  to  $S(\alpha_n)$  is replaced by a contribution to  $S(\gamma)$  of  $M_i''(x_i - c) + M_i'(c - x_{i-1})$ . The difference between these contributions is

$$\begin{aligned} & M_i(x_i - x_{i-1}) - M_i''(x_i - c) - M_i'(c - x_{i-1}) \\ & \leq M(x_i - x_{i-1}) - m(x_i - c) - m(c - x_{i-1}) \\ & = (M - m)(x_i - x_{i-1}) \leq (M - m)\eta. \end{aligned}$$

There are at most  $p - 1$  subintervals of  $\alpha_n$  in which this can happen, and so

$$0 \leq S(\alpha_n) - S(\gamma) \leq (p - 1)(M - m)\eta < \varepsilon/2 \text{ provided } \eta < \varepsilon/(2(p - 1)(M - m)).$$

We therefore have, for all  $n > N$ ,

$$K \leq S(\alpha_n) < S(\gamma) + \varepsilon/2 \leq S(\beta) + \varepsilon/2 < K + \varepsilon.$$

Hence the result.

**Exercise 7** Prove an analogous result for lower sums and lower integrals.

Click [here](#) to see a solution

**Corollary** If we know that  $f$  is integrable, then the above result and that of the exercise apply with  $K$  equal to the value of the integral.

In order to apply this result we need the following theorem.

## The Fundamental Theorem of Calculus: Proof, and an example

### Theorem 17 The Fundamental Theorem of Calculus

If  $f'$  is integrable over  $[a,b]$  then  $\int_a^b f' = f(b) - f(a)$ .

**Proof** Let  $\alpha$  be a subdivision of  $[a,b]$ . By the Mean Value Theorem there is a number  $c_i$  satisfying  $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$ .

Summing over  $I$  gives

$$\sum_{i=1}^n f'(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Therefore  $s(\alpha, f') \leq f(b) - f(a) \leq S(\alpha, f')$ , so  $\int_a^b f' \leq f(b) - f(a) \leq \int_a^b f'$ .

But  $f'$  is integrable and so we have equality throughout.

The above result can be used to evaluate limits involving sums.

**Example 4** Find  $\lim_{n \rightarrow \infty} (1^5 + 2^5 + \dots + n^5)/n^6$ .

$$(1^5 + 2^5 + \dots + n^5)/n^6 = \frac{1}{n} \left( \left(\frac{1}{n}\right)^5 + \left(\frac{2}{n}\right)^5 + \dots + \left(\frac{n}{n}\right)^5 \right).$$

This is an upper sum for  $\int_0^1 x^5$ , with  $\alpha_n = (0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n})$ , i.e.  $x_i = \frac{i}{n}$ . With this

subdivision we have  $M_i = (x_i^5) = \left(\frac{i}{n}\right)^5$ , so  $S(\alpha_n) = \sum_{i=1}^n \left(\frac{i}{n}\right)^5 \frac{1}{n}$ . Now the function  $x^5$  is

continuous and therefore integrable, so

$$\lim_{n \rightarrow \infty} S(\alpha_n) = \int_0^1 x^5 = \left[ \frac{x^6}{6} \right]_0^1 = \frac{1}{6}.$$

### Solution for Exercise 1

**Definition 1** A set  $S$  of real numbers is said to be *bounded below* if

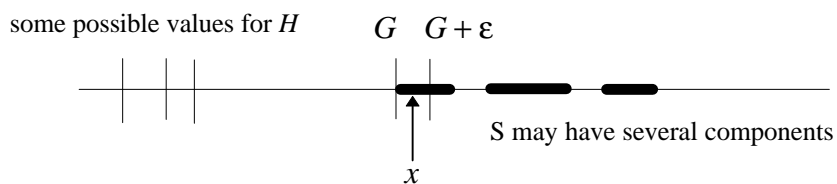
$$\exists H \in \mathbb{R}, \forall x \in S, x \geq H.$$

Any such  $H$  is said to be a *lower bound* for  $S$ .

**Definition 2** Given a set  $S$ , bounded above, a number  $G$  is said to be a *greatest lower bound*, or *infimum*, of  $S$  if

- (a)  $G$  is a lower bound for  $S$ , and
- (b)  $\forall \varepsilon > 0, \exists x \in S, x < G + \varepsilon$ .

It can be shown that when a set has a supremum, it is unique. We write  $G = \inf(S)$ .  
On the number line we have the following picture:



## Solution for Exercise 2

For  $i = 1, \dots, n$ ,  $m \leq M_i \leq M$ , and so

$$\sum_{i=1}^n m(x_i - x_{i-1}) \leq \sum_{i=1}^n M_i(x_i - x_{i-1}) \leq \sum_{i=1}^n M(x_i - x_{i-1}),$$

All the terms in the left hand sum have  $m$  as a common factor, and so the sum can be written as

$$m \sum_{i=1}^n (x_i - x_{i-1}) = m(x_1 - x_0 + x_2 - x_1 + x_3 - x_2 + \dots + x_n - x_{n-1}).$$

The right hand side cancels to give  $m(x_n - x_0) = m(b - a)$ .

Another way of arriving at this result is to realise that we are simply summing the length of all the sub-intervals, giving the total length of the interval, i.e.,  $b - a$ .

Dealing with the top right hand sum in the same way finally gives the result

$$m(b - a) \leq S(\alpha) \leq M(b - a)$$

for upper sums.

### Solution for Exercise 3

**Theorem** If  $a < b < c$  then  ${}^*\int_a^b f + {}^*\int_b^c f = {}^*\int_a^c f$ .

**Proof** Let  $\alpha_1, \alpha_2$  be two arbitrary subdivisions of  $[a, b]$  and  $[b, c]$  respectively.

Together they form a subdivision  $\alpha$  of  $[a, c]$ , and we have

$$S(\alpha_1) + S(\alpha_2) = S(\alpha) \geq {}^*\int_a^c f.$$

[Here we shall give a more detailed argument than in the proof of Theorem 4.]

$$S(\alpha_1) \geq {}^*\int_a^c f - S(\alpha_2).$$

This is true for all  $\alpha_1$ , so the RHS is a lower bound for the set of all upper sums over  $[a, c]$ . Hence  $\inf\{S(\alpha_1)\} \geq {}^*\int_a^c f - S(\alpha_2)$ , i.e.,  ${}^*\int_a^b f \geq {}^*\int_a^c f - S(\alpha_2)$ , which we can rewrite as  $S(\alpha_2) \geq {}^*\int_a^c f - {}^*\int_a^b f$ .

This is true for all  $\alpha_2$ , so a similar argument shows that  ${}^*\int_b^c f \geq {}^*\int_a^c f - {}^*\int_a^b f$ , which gives

$${}^*\int_a^b f + {}^*\int_b^c f \leq {}^*\int_a^c f.$$

Now let  $\beta$  be an arbitrary subdivision of  $[a, c]$ , and let  $\gamma$  be the subdivision formed by adding the point  $b$ . Then  $\gamma$  splits into two subdivisions  $\gamma_1, \gamma_2$  of  $[a, b]$  and  $[b, c]$  respectively, giving

$$S(\beta) \geq S(\gamma) = S(\gamma_1) + S(\gamma_2) \geq {}^*\int_a^b f + {}^*\int_b^c f.$$

Since this is true for all  $\beta$ , we have

$${}^*\int_a^b f + {}^*\int_b^c f \geq {}^*\int_a^c f.$$

Hence we deduce equality.

### Solution for Exercise 4

To prove that  ${}^*\int_a^b f + {}^*\int_a^b g \geq {}^*\int_a^b (f + g)$ .

**Proof** For any interval in a subdivision we have

$$M_i(f) + M_i(g) \geq M_i(f + g). \quad \text{click [here](#) for details}$$

Multiplying by  $(x_i - x_{i-1})$  and summing gives

$$\sum_{i=1}^n M_i(f)(x_i - x_{i-1}) + \sum_{i=1}^n M_i(g)(x_i - x_{i-1}) \geq \sum_{i=1}^n M_i(f + g)(x_i - x_{i-1}),$$

i.e.,

$$S(\alpha, f) + S(\alpha, g) \geq S(\alpha, f + g). \quad (\text{A})$$

The upper integral is the infimum of the set of upper sums, and so we use the definition of the infimum, as follows:

For any  $\varepsilon > 0$  there exist subdivisions  $\beta$  and  $\gamma$  such that

$$S(\beta, f) < {}^*\int_a^b f + \varepsilon/2 \quad \text{and} \quad S(\gamma, g) < {}^*\int_a^b g + \varepsilon/2. \quad (\text{B})$$

We now let  $\alpha$  be the subdivision obtained by including all the points from  $\beta$  and  $\gamma$ , so

$$S(\alpha, f) \leq S(\beta, f) \quad \text{and} \quad S(\alpha, g) \leq S(\gamma, g). \quad (\text{C})$$

Combining the three sets of inequalities (A), (B), (C) above gives

$${}^*\int_a^b f + {}^*\int_a^b g + \varepsilon > S(\alpha, f) + S(\alpha, g) \geq S(\alpha, f + g) \geq {}^*\int_a^b (f + g).$$

This is true for all  $\varepsilon > 0$  and so

$${}^*\int_a^b f + {}^*\int_a^b g \geq {}^*\int_a^b (f + g).$$

**Theorem** Let  $f$  and  $g$  be two bounded functions on an interval  $[a,b]$ . Then

$$(a) \quad m(f) + m(g) \leq m(f + g)$$

$$(b) \quad M(f) + M(g) \geq M(f + g)$$

**Proof**

(a) For all  $x \in [a,b]$ ,  $f(x) + g(x) = (f + g)(x)$ .

Now  $f(x) \geq m(f)$  and  $g(x) \geq m(g)$ , so  $m(f) + m(g) \leq (f + g)(x)$ .

this is true for all  $x \in [a,b]$ , so  $m(f) + m(g)$  is a lower bound for the set of all values of  $(f + g)(x)$  where  $x \in [a,b]$ .

Hence  $m(f) + m(g) \leq m(f + g)$ .

(b) Again from the equation  $f(x) + g(x) = (f + g)(x)$  we deduce that

$$M(f) + M(g) \geq (f + g)(x) \text{ for all } x \in [a,b].$$

So  $M(f) + M(g)$  is an upper bound for the set of all values of  $(f + g)(x)$  where  $x \in [a,b]$ .

Hence  $M(f) + M(g) \geq M(f + g)$ .

### Solution for Exercise 5

Let  $c$  and  $c + h$  both be numbers in the interval  $[a, b]$ , where  $h < 0$ . We then have

$$F(c + h) - F(c) = \int_c^{c+h} f = - \int_{c+h}^c f.$$

**Theorem 2** then tells us that

$$m(c - (c + h)) \leq \int_{c+h}^c f \leq M(c - (c + h)),$$

i.e.,

$$-mh \leq \int_{c+h}^c f \leq -Mh,$$

and so

$$Mh \leq - \int_{c+h}^c f \leq mh,$$

i.e.,

$$Mh \leq F(c + h) - F(c) \leq mh.$$

We deduce that  $\lim_{h \rightarrow 0^-} (F(c + h) - F(c)) = 0$ .

## Solution for Exercise 6

We shall use the notation

$$G(x) = \int_a^x f(t) dt.$$

**Theorem**  $G(x)$  is continuous in  $[a, b]$ .

**Proof** Let  $c$  and  $c+h$  both be numbers in  $[a, b]$ , where  $h > 0$ . We then have

$$G(c+h) - G(c) = \int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f.$$

Using **Theorem 2** enables us to deduce that  $mh \leq G(c+h) - G(c) \leq Mh$ , using the notation introduced at the beginning of this topic, where we restricted discussion to the integration of bounded functions.

It follows that  $\lim_{h \rightarrow 0^+} G(c+h) - G(c) = 0$ .

The argument for  $h < 0$  is exactly similar to that used for lower integrals (click [here](#) to see the details for lower integrals)

**Theorem** If  $f$  is continuous at  $c \in (a, b)$  then  $G$  is differentiable at  $c$ , and  $G'(c) = f(c)$ .

**Proof** Let  $\varepsilon$  be an arbitrary positive number. Because  $f$  is continuous at  $c$  there is a number  $\delta > 0$  such that for  $c - \delta \leq t \leq c + \delta$ ,  $f(c) - \varepsilon < f(t) < f(c) + \varepsilon$ .

Using these bounds and the result of **Theorem 2** tells us that for  $0 < h < \delta$ ,

$$(f(c) - \varepsilon)h < G(c+h) - G(c) < (f(c) + \varepsilon)h, \quad \text{i.e.} \quad \left| \frac{G(c+h) - G(c)}{h} - f(c) \right| < \varepsilon.$$

Hence the result. As in **Theorem 8** the case where  $h$  is negative is left as an exercise.

### Solution for Exercise 7

**Theorem** Let  $f$  be a function bounded on  $[a,b]$ , whose lower integral is  $H$ . Suppose that  $(\alpha_n)$  is a sequence of subdivisions of  $[a,b]$  having the property that the length of the largest subinterval of  $\alpha_n$  tends to zero as  $n \rightarrow \infty$ .

Then  $s(\alpha_n) \rightarrow H$  as  $n \rightarrow \infty$ .

#### Proof

Given  $\varepsilon > 0$ , there is a subdivision  $\beta$  of  $[a,b]$  satisfying  $s(\beta) > H - \varepsilon/2$ . Let  $p$  be the number of sub-intervals (so that  $\beta$  consists of the points  $a, b$  and  $p - 1$  other points).

Given  $\eta > 0$ , there is an integer  $N$  such that for all  $n \geq N$  the length of the largest sub-interval of  $\alpha_n$  is less than  $\eta$ .

Now let  $\gamma$  be the subdivision obtained by adding the points of  $\beta$  to those of  $\alpha_n$ .

We can write  $\alpha_n = (a = x_0, x_1, x_2, \dots, x_r = b)$ .

Using the notation of Theorem 3, if a point  $c$  from  $\beta$  is added in the interval  $(x_{i-1}, x_i)$  then a contribution of  $m_i(x_i - x_{i-1})$  to  $s(\alpha_n)$  is replaced by a contribution to  $s(\gamma)$  of  $m_i''(x_i - c) + m_i'(c - x_{i-1})$ . The difference between these contributions is

$$\begin{aligned} & m_i''(x_i - c) + m_i'(c - x_{i-1}) - m_i(x_i - x_{i-1}) \\ & \leq M(x_i - c) + M(c - x_{i-1}) - m(x_i - x_{i-1}) \\ & = (M - m)(x_i - x_{i-1}) \leq (M - m)\eta. \end{aligned}$$

There are at most  $p - 1$  subintervals of  $\alpha_n$  in which this can happen, and so

$$0 \leq s(\gamma) - s(\alpha_n) \leq (p - 1)(M - m)\eta < \varepsilon/2 \text{ provided } \eta < \varepsilon / (2(p - 1)(M - m)).$$

We therefore have, for all  $n > N$ ,

$$H \geq s(\alpha_n) > s(\gamma) - \varepsilon/2 \geq s(\beta) - \varepsilon/2 > H - \varepsilon.$$

Hence the result.

### Detailed Proof of Theorem 11

**Theorem** If  $a < b < c$  and if  $f$  is integrable over  $[a,b]$  and over  $[b,c]$  then  $f$  is integrable over  $[a,c]$ .

**Proof** We use necessary and sufficient condition for integrability from **Theorem 6**.

Given  $\varepsilon > 0$  there is a subdivision  $\alpha$  of  $[a,b]$  and a subdivision  $\beta$  of  $[b,c]$  such that

$$S(\alpha) - s(\alpha) < \varepsilon/2 \quad \text{and} \quad S(\beta) - s(\beta) < \varepsilon/2.$$

Adding these two inequalities gives

$$S(\alpha) + S(\beta) - (s(\alpha) - s(\beta)) < \varepsilon.$$

Now the points of subdivision  $\alpha$  and the points of  $\beta$  together form a subdivision  $\gamma$  of  $[a,c]$  for which

$$S(\gamma) = S(\alpha) + S(\beta) \quad \text{and} \quad s(\gamma) = s(\alpha) + s(\beta).$$

Therefore

$$S(\gamma) - s(\gamma) < \varepsilon$$

and hence  $f$  is integrable over  $[a,c]$ .

### Corollary

If  $a = t_0 < t_1 < t_2 < \dots < t_n = b$  and if  $f$  is integrable over  $[t_{i-1}, t_i]$ ;  $i = 1, 2, \dots, n$  then  $f$  is integrable over  $[a,b]$ .

**Proof** Use the result of the theorem, and induction on  $n$ .

### Theorem 11

If a bounded function  $f$  has a finite number of discontinuities in  $[a,b]$  then it is integrable over  $[a,b]$ .

### Proof

Suppose that the discontinuities are at the points  $t_0, t_1, t_2, \dots, t_{n-1}$ , where  $a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$ .

Then for  $i = 1, 2, \dots, n$ ,  $f$  is bounded on  $[t_{i-1}, t_i]$  and continuous in  $(t_{i-1}, t_i)$ .

Therefore  $f$  is integrable over  $[t_{i-1}, t_i]$ ;  $i = 1, 2, \dots, n$  and therefore by the corollary is integrable over  $[a,b]$ .

### Calculations for the upper sum for Example 3

Let  $f(x) = x^3$  for  $x \in [1, 2]$ . We consider the subdivision of this interval  $\alpha_n = (1, x, x^2, \dots, x^n = 2)$ , i.e. where  $x = \sqrt[n]{2}$ . We then have

$$m_i = (x^{i-1})^3 \text{ and } M_i = (x^i)^3, \text{ so}$$

$$s(\alpha_n) = \sum_{i=1}^n (x^{i-1})^3 (x^i - x^{i-1});$$

$$S(\alpha_n) = \sum_{i=1}^n (x^i)^3 (x^i - x^{i-1}).$$

We now evaluate the upper sum, which involves GPs.

$$\begin{aligned} S(\alpha_n) &= \sum_{i=1}^n x^{4i} - \sum_{i=1}^n x^{4i-1} = x^4 \left( 1 + x^4 + x^8 + \dots + (x^4)^{n-1} \right) - x^3 \left( 1 + x^4 + x^8 + \dots + (x^4)^{n-1} \right) \\ &= x^3(x-1) \left( \frac{x^{4n} - 1}{x^4 - 1} \right) = x^3 \left( \frac{x^{4n} - 1}{(x+1)(x^2+1)} \right) = \frac{\sqrt[n]{8}(16-1)}{(\sqrt[n]{2}+1)(\sqrt[n]{4}+1)} \rightarrow \frac{15}{4} \text{ as } n \rightarrow \infty. \end{aligned}$$