Difference Equations to **D**ifferential Equations

Section 8.2

Separation of Variables

In the previous section we discussed two methods for approximating the solution of a differential equation

$$\dot{x} = f(x,t)$$

with initial condition $x(t_0) = x_0$. We will now consider, in this section as well as in Sections 8.3 and 8.4, techniques for finding *closed form* solutions for such equations, that is, solutions expressible in terms of the elementary functions of calculus. To do so will require considering different classes of equations depending on the form of the function f. As in ordinary integration, finding a closed form expression for the solution of a differential equation is frequently a difficult, if not impossible, problem which requires us to exploit whatever information we can gain from the form of the function. In this section we will consider a class of equations known as *separable equations* and in Sections 8.3 and 8.4 we will consider linear equations.

We call a differential equation

$$\dot{x} = f(x,t) \tag{8.2.1}$$

with initial condition $x(t_0) = x_0$ *separable*, or say it has *separable variables*, if $f(x,t) = g(x)h(t)$ for some functions g and h, where g depends only on x and h depends only on t. We will assume that g and h are both continuous and hence, in particular, integrable. In that case, (8.2.1) becomes

$$\dot{x} = g(x)h(t) \tag{8.2.2}$$

which implies that

$$\frac{\dot{x}}{g(x)} = h(t) \tag{8.2.3}$$

at all points for which $g(x) \neq 0$. Integrating (8.2.3) from t_0 to t (assuming $g(x(s)) \neq 0$ for all s between t_0 and t), we have

$$\int_{t_0}^{t} \frac{1}{g(x(s))}\ \dot{x}(s)ds = \int_{t_0}^{t} h(s)ds, \tag{8.2.4}$$

where we have used s as the variable of integration so that our answer will be in terms of t. Now the substitution

$$u = x(s)$$
$$du = \dot{x}(s)$$

gives us

$$\int_{t_0}^{t} \frac{1}{g(s)}\ \dot{x}(s)ds = \int_{x(t_0)}^{x(t)} \frac{1}{g(u)}\ du = \int_{x_0}^{x} \frac{1}{g(u)}\ du \tag{8.2.5}$$

for the integral on the left-hand side. Hence, putting (8.2.4) and (8.2.5) together,

$$\int_{x_0}^{x} \frac{1}{g(u)}\, du = \int_{t_0}^{t} h(s)ds. \tag{8.2.6}$$

Thus we can solve an equation with separable variables provided we are able to evaluate both of the integrals in (8.2.6) and then solve the resulting equation for x. The process may break down at either of these final two steps, in which case we must fall back on numerical approximations even though the equation is separable.

Separation of variables If g and h are continuous functions of x and t, respectively, and x satisfies the differential equation

$$\dot{x} = g(x)h(t) \tag{8.2.7}$$

with $x(t_0) = x_0$, then

$$\int_{x_0}^{x} \frac{1}{g(u)}\, du = \int_{t_0}^{t} h(s)ds, \tag{8.2.8}$$

provided $g(u) \neq 0$ for all u between x_0 and x.

Note that this is the same method we used to solve the inhibited growth model equation in Section 6.3.

Example Consider the equation

$$\dot{x} = 0.4x$$

with $x(0) = 100$. This is a separable equation with, in the notation used above, $g(x) = x$ and $h(t) = 0.4$. (Note that the choices for g and h are not unique.) Using (8.2.8), we have

$$\int_{100}^{x} \frac{1}{u}\, du = \int_{0}^{t} 0.4ds,$$

Now, assuming $x > 0$,

$$\int_{100}^{x} \frac{1}{u}\, du = \log(u)\Big|_{100}^{x} = \log(x) - \log(1000) = \log\left(\frac{x}{100}\right),$$

and

$$\int_{0}^{t} 0.4ds = 0.4s\Big|_{0}^{t} = 0.4t.$$

Hence we have

$$\log\left(\frac{x}{100}\right) = 0.4t,$$

from which we obtain

$$\frac{x}{100} = e^{0.4t}$$

and, finally,

$$x = 100e^{0.4t}.$$

Note that this is the solution we should expect from our study of equations of this form in Sections 6.1 and 6.3.

Example Consider the equation

$$\dot{y} = -2yt \tag{8.2.9}$$

with $y(0) = y_0 \neq 0$. This is a separable equation with, in the notation used above, $g(y) = y$ and $h(t) = -2t$. Using (8.2.8), we have

$$\int_{y_0}^{y} \frac{1}{u}\, du = -\int_0^t 2sds.$$

Now

$$\int_{y_0}^{y} \frac{1}{u}\, du = \log|u|\Big|_{y_0}^{y} = \log|y| - \log|y_0| = \log\left|\frac{y}{y_0}\right|$$

and

$$-\int_0^t 2sds = -s^2\Big|_0^t = -t^2.$$

Hence we have

$$\log\left|\frac{y}{y_0}\right| = -t^2,$$

from which it follows that

$$\left|\frac{y}{y_0}\right| = e^{-t^2}.$$

Now $e^{-t^2} > 0$ for all t, so $y(t)$ is never 0. Since y is continuous (which follows from our assumption that it is differentiable), this means that either $y(t) > 0$ for all t or $y(t) < 0$ for all t. Since $y(0) = y_0$, $y(t) > 0$ for all t if $y_0 > 0$ and $y(t) < 0$ for all t if $y_0 < 0$. In either case,

$$\frac{y(t)}{y_0} > 0$$

for all t, so

$$\left|\frac{y}{y_0}\right| = \frac{y}{y_0}.$$

Hence we have

$$\frac{y}{y_0} = e^{-t^2},$$

or

$$y = y_0 e^{-t^2}. \tag{8.2.10}$$

Note that (8.2.10) also specifies a solution of (8.2.9) when $y_0 = 0$, namely, the solution $y(t) = 0$ for all t. By leaving the value of y_0 unspecified, we have found the general form of all possible solutions for the equation. We call the family of all possible solutions given by (8.2.10) the *general solution* of the equation (8.2.9). Any solution obtained by specifying

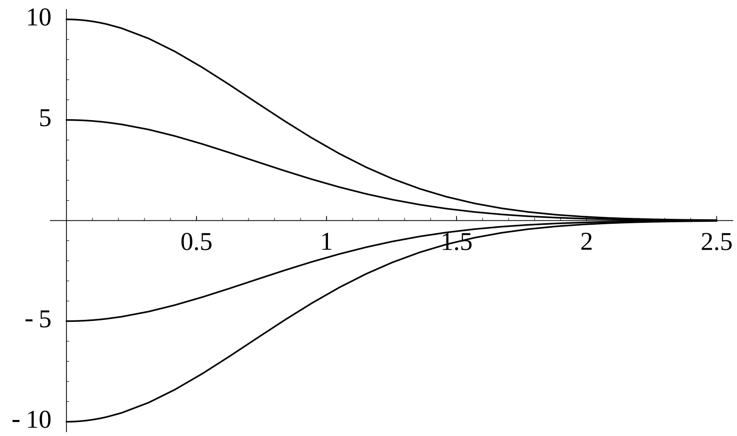

Figure 8.2.1 Four particular solutions of $\dot{y} = -2yt$

a value of y_0, say, for example, $y_0 = 10$, is called a *particular solution* of the equation. Figure 8.2.1 shows the graphs of four particular solutions for this equation.

As noted in the first example, the choices for g and h are not unique. For example, in the second example we could just as well have taken $g(y) = 2y$ and $h(t) = t$. However, one should attempt to choose g and h in such a way that the subsequent steps in the solution are as simple as possible.

Example Consider the equation

$$\dot{x} = -\frac{t}{x}$$

with $x(0) = x_0 \neq 0$. Separating the variables, we have

$$\int_{x_0}^{x} u du = -\int_0^t s ds.$$

Now

$$\int_{x_0}^{x} u du = u^2\Big|_{x_0}^{x} = x^2 - x_0^2$$

and

$$-\int_0^t s ds = -s^2\Big|_0^t = -t^2,$$

and so

$$x^2 - x_0^2 = -t^2,$$

or

$$x^2 + t^2 = x_0^2.$$

This equation implicitly defines x as a function of t. Indeed, from this equation we can see that the graph of x is part of circle of radius x_0 centered at the origin. Solving explicitly for x, we have

$$x = \sqrt{x_0^2 - t^2}$$

if $x_0 > 0$ and

$$x = -\sqrt{x_0^2 - t^2}$$

if $x_0 < 0$. Note that x is only defined for $-x_0 < t < x_0$.

Example In Section 8.1 we considered the equation

$$\dot{v} = -g - \frac{k}{m}v,$$

with $v(0) = 0$, as a model for the velocity of an object in free fall near the surface of the earth when the force due to air resistance is proportional to velocity. Here v is the velocity of the object, g, as usual, is 32 feet per second per second or 9.8 meters per second per second, m is the mass of the object, and $k > 0$ is a constant which depends on the air resistance of the particular object. If we write this equation in the form

$$\dot{v} = -g\left(1 + \frac{k}{gm}v\right) \tag{8.2.11}$$

and separate variables, using

$$f(v) = 1 + \frac{k}{gm}v$$

and

$$h(t) = -g,$$

then we have

$$\int_0^v \frac{1}{1 + \frac{k}{gm}u}\,du = -\int_0^t g ds.$$

Now

$$\int_0^v \frac{1}{1 + \frac{k}{gm}u}\,du = \frac{gm}{k}\log\left|1 + \frac{k}{gm}u\right|\Big|_0^v = \frac{gm}{k}\log\left|1 + \frac{k}{gm}v\right|$$

and

$$-\int_0^t g ds = -gs\Big|_0^t = -gt,$$

so

$$\frac{gm}{k}\log\left|1 + \frac{k}{gm}v\right| = -gt.$$

Hence

$$\log\left|1 + \frac{k}{gm}v\right| = -\frac{kt}{m},$$

from which it follows that

$$\left|1 + \frac{k}{gm}v\right| = e^{-\frac{kt}{m}}.$$

Thus either

$$1 + \frac{k}{gm}v = e^{-\frac{kt}{m}}$$

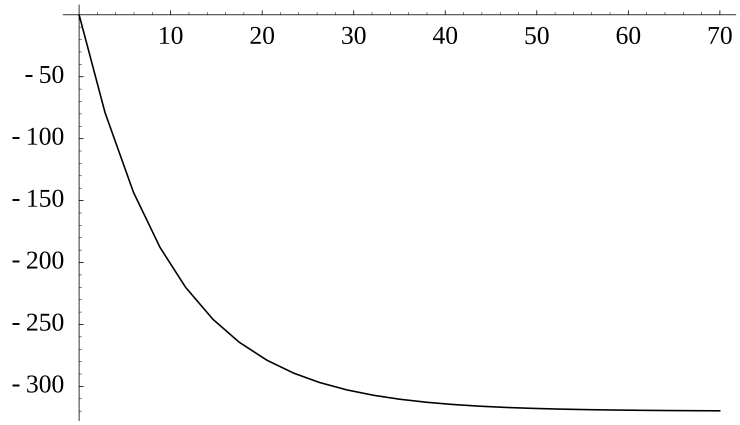

Figure 8.2.2 Graph of $v(t) = -320(1 - e^{-0.1t})$

or

$$1 + \frac{k}{gm}v = -e^{-\frac{kt}{m}}.$$

That is, either

$$v = -\frac{gm}{k}(1 - e^{-\frac{kt}{m}}).$$

or

$$v = -\frac{gm}{k}(1 + e^{-\frac{kt}{m}}).$$

Since our initial condition requires that $v(0) = 0$, we must have

$$v = -\frac{gm}{k}(1 - e^{-\frac{kt}{m}}).$$

Hence we now have a closed form solution for this model of free fall, whereas in the previous section we could only compute a numerical approximation. Notice that one advantage of the closed form solution is that we did not have to specify values for the parameters k and m before finding the solution; as a result, we may now easily compute v for any specified values of k and m. For example, using $\frac{k}{m} = 0.1$ and $g = 32$ as in our example in Section 8.1, we obtain a plot of v as shown in Figure 8.2.2. You should compare this with the graph of our numerical solution in Figure 8.1.3. Also, the closed form solution allows us to compute

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -\frac{gm}{k}(1 - e^{-\frac{kt}{m}}) = -\frac{gm}{k}, \tag{8.2.12}$$

showing that an object falling according to this model has a terminal velocity, as we suspected from our numerical work in Section 8.1. Moreover, (8.2.12) gives us a general expression for this velocity. For our example, $\frac{k}{m} = 0.1$ and $g = 32$ give us a terminal velocity of

$$-\frac{gm}{k} = -320 \text{ feet per second.}$$

Problems

1. Solve each of the following differential equations using the given initial condition.
   (a) $\dot{x} = -0.9x$, $x(0) = 75$
   (b) $\dot{x} = x^2$, $x(0) = 10$
   (c) $\dot{y} = \frac{t}{y}$, $y(0) = 5$
   (d) $\dot{w} = \frac{w}{t}$, $w(1) = 1$
   (e) $\dot{x} = \frac{t}{x + tx}$, $x(0) = 4$
   (f) $\dot{y} = 1 + y^2$, $y(0) = 0$
   (g) $\dot{x} = x(1 - x)$, $x(0) = 0.2$
2. (a) Solve the differential equation $\dot{x} = -x^2\mathrm{t}$, $x(0) = x_0 \neq 0$.
   (b) Graph x on the interval $[5, 5]$ for $x_0 = 2$, $x_0 = 5$, and $x_0 = 10$. Are the graphs similar?
   (c) What is the domain of x if $x_0 > 0$? What is the domain of x if $x_0 < 0$?
   (d) Graph x for $x_0 = -1$ and $x_0 = 1$. Are the graphs similar?
3. (a) A curve is defined so that whenever (x_0, y_0), with $y_0 \neq 0$, is a point on the curve,
$$\left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = -\frac{ax_0}{by_0},$$
   where $a > 0$ and $b > 0$ are constants. Show that the curve must be an ellipse. Under what conditions is the curve a circle?
   (b) A curve is defined so that whenever (x_0, y_0), with $y_0 \neq 0$, is a point on the curve,
$$\left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = \frac{ax_0}{by_0},$$
   where $a > 0$ and $b > 0$ are constants. Show that the curve must be a hyperbola.
4. In Chapter 6 we considered the consequences of the population growth model
$$\dot{x} = kx,$$
   with $x(0) = x_0$, where $x(t)$ represents the size of some population at time t and $k > 0$ is a constant which depends on the rate at which the population is growing. In this problem we will see what happens if $\dot{x}$ is proportional, not to x, but to some power of x. That is, consider the model
$$\dot{x} = kx^b, \tag{8.2.13}$$
   with $x(0) = x_0$ and $b > 0$ a constant.
   (a) Solve (8.2.13) when $b = 2$ and show that
$$\lim_{t \to \frac{1}{kx_0}^-} x(t) = \infty.$$
   Plot x for $x_0 = 50$ and $k = 0.001$, $k = 0.01$, and $k = 0.02$.

(b) Solve (8.2.13) when $b > 1$. Find c such that

$$\lim_{t \to c^-} x(t) = \infty.$$

Plot x for $x_0 = 50$, $k = 0.01$, and $b = 1.5$, $b = 1.2$, and $b = 1.01$.

(c) Solve (8.2.13) when $b = 0.5$. Show that x is a quadratic polynomial and

$$\lim_{t \to \infty} x(t) = \infty.$$

Plot x for $x_0 = 50$ and $k = 0.01$, $k = 0.02$, and $k = 0.05$.

(d) Solve (8.2.13) when $0 < b < 1$ and show that

$$\lim_{t \to \infty} x(t) = \infty.$$

Plot x for $x_0 = 50$, $k = 0.01$, and $b = 0.2$, $b = 0.4$, and $b = 0.9$.

(e) Compare the rates of growth for $0 < b < 1$, $b = 1$, and $b > 1$. Which model leads to the slowest population growth? Which model leads to the most rapid population growth? Why is the case $b > 1$ sometimes referred to as the *doomsday model*?

5. Suppose the force due to air resistance acting on a falling body of mass m is proportional to the square of the velocity v.

(a) Explain why v satisfies the differential equation

$$\dot{v} = -g + \frac{k}{m}v^2,$$

where $k > 0$ is a constant.

(b) Assuming $v(0) = 0$, solve the equation in (a) for v.

(c) Show that the terminal velocity of the object is $-\sqrt{\frac{mg}{k}}$.

(d) Plot v over the interval $[0, 20]$ using $g = 32$ and $\frac{k}{m} = 0.01$. Compare this plot with the plot of the numerical solution found in Problem 8 of Section 8.1.

6. In Section 1.4 we discussed the discrete time version of Newton's law of cooling. Briefly, this law says if an object with an initial temperature of T_0 is placed in an environment which is held at a constant temperature S, then the rate of change of the temperature T of the object is proportional to the difference between T and S. In terms of differential equations, this says that T must satisfy the equation

$$\dot{T} = k(T - S)$$

for some constant k.

(a) Show that

$$T = S + (T_0 - S)e^{kt}$$

and verify that

$$\lim_{t \to \infty} T(t) = S.$$

(b) A cup of coffee, initially at a temperature of 115°F, is placed on a table in a room held at a constant temperature of 72°F. If after five minutes the coffee has cooled to 105°F, what is the temperature of the coffee after 20 minutes? How long will it take the coffee to cool to 80°F? Graph T.

(c) A glass of lemonade, initially at a temperature of 40°F, is placed on a table in a room held at a constant temperature of 75°F. If after 10 minutes the lemonade has warmed to 48°F, what is the temperature of the lemonade after 30 minutes? How long will it take the lemonade to warm to 65°F? Graph T.

(d) A cup of coffee, initially at a temperature of 110°F, is placed on a table in a room. After five minutes the coffee has cooled to 100°F and after ten minutes the coffee has cooled to 92°F. What is the temperature of the room?